Investigation of the Properties of Strongly Interacting Particles in the Nambu-Jona-Lasinio Model

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Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

Momchil N. Naydenov
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Abstract

The focus of this work is the effective description of low energy strong interactions in the Nambu-Jona-Lasinio model. We start with a simple U(1) symmetric theory with a massless quark. This model was extended to SU(2) symmetry with a massless quark and a novel mass relation between the meson excitations was derived. We further consider a U(1) model with a massive quark. The massive U(1) model reproduces the same result as for the massless case. Another problem we considered was the isotopic symmetry breaking. It was shown that within the U(2) scalar Nambu-Jona-Lasinio model we cannot describe the difference between the masses of the up and down quark. After extending to a U(3) model we concluded that the Nambu-Jona-Lasinio model can be a possible description for the existence of isotopic symmetry in Nature. All these results are the essence of the work we carried out during my PhD degree, present in the current thesis.

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Chapter 1

Introduction

The Standard Model is the most successful tool for describing fundamental particles and their interactions [1]. It is a non-abelian gauge theory with symmetry group $SU(3) \times SU(2) \times U(1)$. The strong interaction governing the binding of quarks inside mesons and baryons, as well as the neutrons and protons in the atomic nucleus, is mathematically described by the $SU(3)_C$ gauge group. There exist 8 bosons, the gluons, mediating the strong interaction among coloured particles - the 6 quark flavours. One of the greatest unsolved problems in modern physics is why we cannot observe a free gluon or quark, i.e. there is *confinement* between all coloured particles [2].

The electromagnetic force acting between electrically charged particles is unified with the weak interaction, governing the nuclear decay, into a single electroweak theory described by the group $SU(2)_W \times U(1)_Y$, known as the Glashow-Weinberg-Salam model. The weak interaction is mediated by the $W^{\pm}$ and Z bosons and the electromagnetic interaction - by the photon. For their works [3–5] on the unified electroweak theory they receive the Nobel Prize in physics in 1979.

In gauge field theories Lagrangians obey gauge (local) symmetries but the lowest energy state may undergo spontaneous symmetry breaking. This is a very fundamental effect present in all physical theories due to which we can describe the masses of the different vacuum excitation modes. In the theory of electroweak interactions the spontaneous symmetry breaking of $SU(2) \times U(1)$ leads to the generation of the masses of $W^{\pm}$ and Z boson, fermions, and most importantly - the existence of the Higgs boson is predicted. The Higgs field is spread throughout space and breaks certain symmetries and the interaction between particles and the Higgs field gives rise to the mass of those particles through a process called Higgs mechanism. The vacuum excitation of the Higgs field is the Higgs boson. The theory behind this process was mainly developed simultaneously by R. Brout and F. Englert [6], P. Higgs [7] and G. Guralnik, C. Hagen and T. Kibble [8]. The discovery of the Higgs boson at the LHC on 4 July 2012 led to the 2013 Nobel prize being jointly awarded to Peter Higgs and Francois Englert.

The other ingredient of the Standard Model (SM) is the quantum chromodynamics (QCD). It is a chiral theory and the breaking of its symmetry leads to the correct description of baryons and mesons. In the limited case of QCD theory with three massless quarks we have a theory with global $SU(3) \times SU(3)$ chiral symmetry. Under symmetry breaking 8 massless Nambu-Goldstone bosons are generated. The explicit introduction of quark mass immediately breaks chiral symmetry. Heavier quarks (c, b, t) cannot be trivially described by the chiral theory as their masses are much heavier than the masses of u, d and s quark and cannot be treated as small perturbations. The correct description of chiral symmetry breaking was first done by Yoichiro Nambu and Giovanni Jona-Lasinio [9, 10] in 1961 for which Nambu was awarded half of the Nobel prize in 2008. The determination of the low energy structure of the Green's function in QCD is thoroughly described in [11]. The works of D. Gross, F. Wilczek and H. Politzer [12, 13] predict the fundamental property of QCD - asymptotic freedom, for which they share the 2004 Nobel Prize. One consequence is that QCD is renormalizable at high energies and for low energies we need to adopt an effective field theory. Such theory is the chiral perturbation theory which allows to study the behaviour of strong interacting particles at low energies. It must contains all symmetries of QCD and have a firm link to it, as it is stated by Weinberg [14]. At low energies the strong interaction leads to confinement of quarks and colourless object are formed - hadrons. A physical effective field theory needs to meet certain criteria. If we assume that the results of QCD need to be matched with the low energy theory, then the criteria are that the low energy degrees of freedom and symmetries are the same as for high energy [15].

In this thesis we will investigate the consequences of a type of a chiral theory - the Nambu-Jona-Lasinio model (NJL). In it, the low-energy strong interaction model is developed by analogy with the superconductivity model by Bardeen, Cooper and Schriefer [16] where we have interaction between electrons and phonons (bosons). In the case of a quantum field theory we have effective Yukawa coupling between fermions and mesons. To obtain the dynamics of the particles we calculate all Feynman diagrams up to 1-loop level. An important property of the NJL model is that there is no colour confinement and quarks are free particles. It was also discovered [17] that the model is renormalizable and in a nonlinear theory with fermions, scalars, pseudoscalars, vectors and pseudovectors the scalar/pseudoscalar parts interact with the vector/pseudovector parts like a complex Higgs-like field with gauge fields.

Therefore, the dynamical breaking of chiral symmetry gives mass to meson excitation modes. The spontaneous breaking of chiral symmetry in the strong interaction leads to similar results as the Higgs mechanism, so that quarks, mesons baryons acquire mass [18][19][20].

A natural extension to the vector sector in the NJL model without is the introduction of tensor currents. We develop theories which are local and Lorentz-invariant without including effective terms with derivatives of the fields. There have been previously discussions on that matter, in fact T. Eguchi mentions that possibility in the paper cited here, but finds it "uninteresting". Other papers [21, 22] discuss the possibility of introducing tensor currents and even derive an equation relating the masses of ρ, π, B and σ mesons. Mainly for historical reasons tensor interactions have been skipped.

Here we give the motivation for why tensor currents can be introduced so that chiral symmetry in the NJL model is not broken. The paper by T. Eguchi finishes by showing the difficulty regarding tensor currents. The only possible Lorentz invariant local interaction is $\bar{\psi}\sigma_{\mu\nu}\psi$. under axial rotation at an angle α this current transforms as

$$\bar{\psi}\sigma_{\mu\nu}\psi \to \cos 2\alpha \bar{\psi}\sigma_{\mu\nu}\psi + \sin 2\alpha \bar{\psi}i\gamma^5\sigma_{\mu\nu}\psi \tag{1.1}$$

In order for the tensor current to be invariant it should be introduced as

$$\left(\bar{\psi}\sigma_{\mu\nu}\psi\right)^2 + \left(\bar{\psi}i\gamma^5\sigma_{\mu\nu}\psi\right)^2 \tag{1.2}$$

which however is equivalently equal to 0. We think that this is the main reason for omitting the introduction of tensor mesons. Apparently the introduction of tensor currents in a chirally invariant manner is not possible. In this thesis we included tensor currents by conducting dynamical analysis of the associated collective modes. As a result of the existence of fourth-order interactions with scalars (box Feynman diagrams to be later derived), a spontaneous breaking of the chiral symmetry occurs like in the Higgs mechanism, but it turns out that it is dynamic for tensor mesons. As a results, interactions with a massive scalar meson gives rise to massive excitation modes.

The idea which is exploited to a great detail is the introduction of tensor currents in the NJL model, obeying $U(1)$, $SU(2)$ and $SU(3)$ symmetry independently. By starting from effective Yukawa couplings between scalars and quarks we develop a full dynamical description of all possible meson excitations which leads to novel mass relations between the mesons [23]. We show that these relations and the experimental data have a good overlap both in the case of massless quarks and in the case when we explicitly break chiral symmetry by introducing a non-zero quark mass. The predictions of the NJL model for the chiral symmetry breaking and the pion decays is investigated [24]. A prediction of the mass of the a_1 meson is made in [25] on the basis of the model including tensor excitations. The combination between the theoretical description in this thesis which include tensor mesons together with the known models in low energy strong interaction can fill in a niche in the accuracy of meson masses calculations. These results which are beyond the SM calculations give a better understanding of the meson physics.

Moreover, we show that the NJL model with two quarks conserves the isotopic symmetry between the u and d quark but the SU(3) model predicts the splitting between the masses of the u, d and s quark where the quark mass differences is governed by a single parameter. The theory presented steps on and expands the works of M. Chizhov [26], A. Osipov [27] and U. Meissner [28]. The work which is summarized in the following pages is mainly theoretical but we are also excited about the experimental realization of the models.

This thesis is organized as follows: the second chapter summarizes the most widely used theory and examples in this thesis, which was also covered in the courses I had to go through during my "theoretical foundation" in the beginning of the PhD degree. Then we derive the main results for an NJL model with $SU(2)$ symmetry and a massless quark. The idea was to calculate all Feynman diagrams up to 1-loop, with 2, 3, and 4 external lines. We obtained an effective Lagrangian, which we were hoping to be able to write in a compact form, but turned out to be impossible. Then we included tensor interactions and finally, we obtained experimentally measurable result for a connection between the meson excitation masses. Once we had the results from the literature for the case of $U(1)$ symmetry with a massless quark, we had the idea of trying to recover the same results by including a massive quark, or even get a more accurate result. It turned out that the mass ratio between the meson excitations is the same with a massive quark, compared to the case of a massless quark. The final chapter is dedicated to the isotopic symmetry, we manage to show that we can potentially use the NJL model as an effective description for the emergence of this symmetry in Nature, which leads to the equivalence, for example, in the mass of the u and d quark, the mass of the proton and the neutron and so on. Furthermore, we deal with the scalar models with $SU(2)$ and $SU(3)$ symmetry, where we try to obtain the mass spectrum of the mesons. There is a long-standing problem in the case of $SU(3)$ symmetry with the so-called inverted mass spectrum, where the mass of κ mesons is smaller than the mass of the a mesons and the f_0 meson. We show that in the case of a massless quark model we cannot obtain the spectrum corresponding to reality, since the κ mesons in this model play the role of the four Goldstone bosons, which are massless. It is known that κ mesons are massive, so $SU(3)$ NJL model with a massless quark is not a physical theory. A problem which would be the topic of a future work will be a complete development of an $SU(3)$ model with a massive u, d and s quark, which will explicitly break the symmetry in the theory. We hope that such model will correctly describe the inverted meson masses spectrum.

Chapter 2

Background theory and examples

2.1 Wick pairing and application to electron-positron scattering

The main sources for the theoretical background in quantum field theory in this thesis are the book by N. Bogoliubov and D. Shirkov-"Introduction to the Theory of Quantized Fields" [29], "Quantum Field Theory" by M. Peskin and D. Schroeder [30] and "Gauge Theory of Elementary Particle Physics" [19] by Ta-Pei Cheng and Ling-Fong Li. We start by introducing the scattering amplitude as

$$S = T\left[\exp\left(i\int \mathscr{L}(x)dx\right)\right], \tag{2.1}$$

where $\mathscr{L}(x)$ is the Lagrangian of the system and T stands for time-ordered product. We expand the exponent in Taylor series and depending on the order of the process that we are interested in we take the corresponding term from the series. In perturbation theory

$$S = T\left[1 + i\int \mathscr{L}(x)dx + \frac{i^2}{2}\int \mathscr{L}(x)\mathscr{L}(y)dxdy + ...\right] \tag{2.2}$$

and we denote

$$S_n = i^n T\left[\mathscr{L}(x_1)...\mathscr{L}(x_n)\right] \tag{2.3}$$

Wick's theorem states that the T-product is the normally ordered product and all possible pairings. For the different fields the pairings are the causal Green's functions, which we will list below for scalars, photons, vector fields and spinor fields.

$$\phi(x)\phi(y) = -i\int \frac{e^{ik(x-y)}}{m^2-k^2-i\varepsilon}dk \tag{2.4}$$

$$A_\mu(x)A_\nu(y) = -ig_{\mu\nu}\int \frac{e^{ik(x-y)}}{k^2+i\varepsilon}dk \tag{2.5}$$

$$U_\mu(x)U_\nu(y) = i\int \left[\frac{g_{\mu\nu}-\frac{k_\mu k_\nu}{m^2}}{m^2-k^2-i\varepsilon}\right] e^{ik(x-y)}dk \tag{2.6}$$

$$\psi_\mu(x)\bar{\psi}_\nu(y) = -i\int \frac{(m-\hat{p})_{\mu\nu}}{m^2-p^2-i\varepsilon}e^{ip(x-y)}dp \tag{2.7}$$

Also

$$T\left[J^\mu(x),J^\nu(y)\right] = T\left[:\bar{\Psi}(x)\gamma^\mu\Psi(x)::\bar{\Psi}(y)\gamma^\nu\Psi(y):\right] =$$

$$=:\bar{\Psi}(x)\gamma^\mu\Psi(x)\bar{\Psi}(y)\gamma^\nu\Psi(y): -i:\bar{\Psi}(y)\gamma^\nu S^c(y-x)\gamma^\mu\Psi(x): -i:\bar{\Psi}(x)\gamma^\mu S^c(x-y)\gamma^\nu\Psi(y):$$

$$+\,\mathrm{Tr}\left[S^c(x-y)\gamma^\nu S^c(y-x)\gamma^\mu\right] \tag{2.8}$$

As a simple example we take the electron-positron interaction. We will see that two channels are allowed - scattering and annihilation. The second order processes in e in quantum electrodynamics (QED) are described by the scattering matrix

$$S_2(x,y) = -e^2T\left[:J^\mu(x)J^\nu(y):\right]T\left[:A_\mu(x)A_\nu(y):\right] \tag{2.9}$$

Here $A_\mu(x)$ is the electromagnetic four-potential in Minkowski space-time and using the definition of time-ordered product we have

$$T\left[:A_\mu(x)A_\nu(y):\right] =:A_\mu(x)A_\nu(y): +ig_{\mu\nu}D_0^c(x-y), \tag{2.10}$$

where $D_0^c(x-y)$ is the photon propagator. Therefore, the matrix becomes

$$S_2(x,y) = -e^2\Big[:J^\mu(x)A_\mu(x)J^\nu(y)A_\nu(y): +i:\bar{\Psi}(x)\gamma^\mu\Psi(x)\bar{\Psi}(y)\gamma_\mu\Psi(y):D_0^c(x-y)-$$

$$-i:\bar{\Psi}(y)\gamma^\mu A_\nu(y)S^c(x-y)\gamma^\mu A_\mu(x): + :\bar{\Psi}(y)\gamma_\mu S^c(y-x)\gamma^\mu\Psi(x): -$$

$$-i : \bar{\Psi}(x)\gamma^{\mu}A_{\mu}(x)S^{c}(x-y)\gamma^{\nu}A_{\nu}(y)\Psi(y) : + : \bar{\Psi}(x)\gamma^{\mu}S^{c}(x-y)\gamma_{\mu}\Psi(y) : D_0^c(x-y)+$$

$$\mathrm{Tr}\left[: S^{c}(x-y)\gamma^{\nu}A_{\nu}(y)S^{c}(y-x)\gamma^{\mu}A_{\mu} :\right] + iD_0^c(x-y)\mathrm{Tr}\left[S^{c}(x-y)\gamma_{\mu}S^{c}(x-y)\gamma^{\mu}\right]\Big] \quad (2.11)$$

where each term corresponds to a Feynman diagram in coordinate space. We are particularly interested in the process below

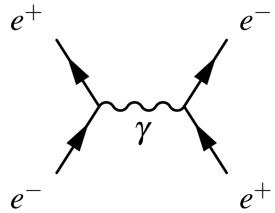


Fig. 2.1 General electron-positron diagram, where the interaction is mediated by a photon.

For this process the in and out states can be defined as

$$|\mathrm{in}\rangle = a_{\nu_1}^{*+}(p_1)a_{\nu_2}^{+}(p_2)|\Phi_0\rangle \quad (2.12)$$

$$\langle\mathrm{out}| = \langle\Phi_0^{\dagger}|a_{\nu_2'}^{*-}(p_2')a_{\nu_1'}^{-}(p_1') \quad (2.13)$$

Therefore, the matrix element for this process is

$$\begin{gathered} M = \frac{-ie^2}{2}\langle\Phi_0^{\dagger}|a_{\nu_2'}^{*-}(p_2')a_{\nu_1'}^{-}(p_1') \\ \int dxdy\bar{\Psi}(x)\gamma^{\mu}\Psi(x)\bar{\Psi}(y)\gamma_{\mu}\Psi(y)D_0^c(x-y)a_{\nu_1}^{*+}(p_1)a_{\nu_2}^{+}(p_2)|\Phi_0\rangle \end{gathered} \quad (2.14)$$

We use Fourier decomposition for the spinor functions to obtain that

$$\Psi(x) = \sum_{\nu}\int dpe^{ipx}v_{\sigma}^{\nu+}a_{\nu}^{+}(p) + e^{-ipx}v_{\sigma}^{\nu-}(p)a_{\nu}^{-}(p) \quad (2.15)$$

$$\bar{\Psi}(x) = \sum_{\nu}\int dpe^{ipx}\bar{v}_{\sigma}^{\nu+}a_{\nu}^{*+}(p) + e^{-ipx}\bar{v}_{\sigma}^{\nu-}(p)a_{\nu}^{*-}(p) \quad (2.16)$$

Applying Wick's theorem we get four matrix elements which reduce to two, corresponding to s and t-channel. Writing all contractions explicitly gives

$$a^-_{\nu'_1}(p'_1)\bar{\Psi}(x) = \sum_\alpha \int dq_1 \Big[-e^{-iq_1x}\bar{v}^\alpha_\sigma(q_1)a^{*+}_\alpha a^-_{\nu'_1}(p'_1) + e^{ipx}\bar{v}^{\alpha+}_\sigma(q_1)\delta(q_1-p'_1)\delta_{\alpha\nu'_1} + \\ +e^{-iq_1x}\bar{v}^{\alpha-}_\sigma(q_1)a^-_{\nu'_1}(p'_1)a^{*-}_\alpha(q_1)\Big] \tag{2.17}$$

$$a^{*-}_{\nu'_2}(p'_2)\Psi(x) = \sum_\beta \int dq_2 \Big[-e^{iq_2x}v^{\beta+}_\rho(q_2)a^+_\beta(q_2)a^{*-}_{\nu'_2}(p'_2) + \\ +e^{iq_2x}v^{\beta+}_\rho(q_2)\delta(q_2-p'_2)\delta_{\beta\nu'_2} + e^{-iq_2x}v^{\beta-}_\rho(q_2)a^{*-}_{\nu'_2}(p'_2)a^{*-}_\beta(q_2)\Big] \tag{2.18}$$

$$\Psi(y)a^{*+}_\nu(p_2) = \sum_\delta \int dq_4 [e^{iq_4y}v^{\delta+}_\chi(q_4)a^+_\delta(q_4)a^{*+}_{\nu'_1}(p_1) - \\ -e^{iq_4y}v^{\delta-}_\chi(q_4)a^{*+}_{\nu_1}(p_1)a^-_\delta(q_4) + e^{-iq_4y}v^{\delta-}_\chi(q_4)\delta(p_1-q_4)] \tag{2.19}$$

Therefore,

$$M_1 = \frac{e^2}{2}\int dxdydq_1dq_2dq_3dq_4 \sum_{\alpha\beta\gamma\delta} \langle\Phi^\dagger_0| \Big[-e^{iq_1x}\bar{v}_\sigma(q_1)a^{*+}_\alpha(q_1)a^-_{\nu_1}(p'_1) + \\ +e^{iq_1x}\bar{v}^{\alpha+}_\sigma(q_1)\delta(q_1-p'_1)\delta_{\alpha\nu'_1} + e^{-iq_1x}\bar{v}^{\alpha-}_\sigma(q_1)a^-_{\nu'_1}(p'_1)a^{*-}_\alpha(q_1)\Big]\gamma^\mu \\ \Big[-e^{iq_2x}v^{\beta+}_\rho(q_2)a^+_\beta(q_2)a^{*-}_{\nu'_2}(p'_2) + e^{iq_2x}v^{\beta+}_\rho(q_2)\delta(q_2-p'_2)\delta_{\beta\nu'_2} + \\ e^{-iq_2x}v^{\beta-}_\rho(q_2)a^{*-}_{\nu'_2}(p'_2)a^{*-}_\beta(q_2)\Big] \\ \Big[e^{iq_4y}v^{\delta+}_\chi(q_4)a^+_\delta(q_4)a^{*+}_{\nu_2}(p_1) - e^{-iq_4y}v^{\delta-}_\chi(q_4)a^{*+}_{\nu_1}(p_1)a^-_\delta(q_4) + \\ e^{-iq_4y}v^{\delta-}_\chi(q_4)\delta(p_1-q_4)\delta_{\delta\nu_2}\Big]\gamma^\mu\Big[e^{iq_3y}\bar{v}^{\gamma+}_\omega(q_3)a^+_{\nu_2}(p_2) \\ -e^{-iq_3y}\bar{v}^{\gamma-}_\omega(q_3)a^+_{\nu_2}(p_2)a^{*-}_\gamma(q_3) + e^{-iq_3y}\bar{v}^{\gamma-}_\omega(q_3)\delta(q_3-p_2)\delta_{\gamma\nu_2}\Big]D^c_0(x-y)|\Phi_0\rangle \tag{2.20}$$

Opening all above brackets and sandwiching between the vacuum states leaves only one term, and after integration the matrix element becomes

$$M_1 = -\frac{ie^2}{2}\bar{v}^{\nu'_1+}_\sigma(p'_2)\gamma^\mu v^{\nu'_2+}_\rho(p'_2)\frac{1}{(p_1+p_2)^2}v^{\nu_1-}_\chi(p_1)\gamma_\mu\bar{v}^{\nu_2-}_\omega(p_2) \tag{2.21}$$

Following the same steps for the other channels' matrix elements gives

$$M_2 = -\frac{1}{2} ie^2 \bar{v}_{\sigma}^{\nu_2 -}(p_2)\gamma^{\mu} v_{\rho}^{\nu_2' +}(p_2') \frac{1}{(p_2 - p_1')^2} \bar{v}_{\omega}^{\nu_1' +}(p_1')\gamma_{\mu} v_{\chi}^{\nu_1 -}(p_1) \tag{2.22}$$

The two channels are s-channel and t-channel, where the definition of the Mandelstam variables are $s = (p_1 + p_2)^2$ and $t = (p_1 - p_1')^2$.

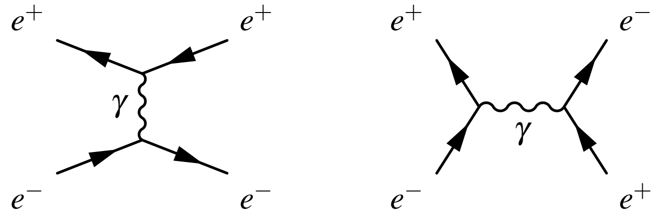


Fig. 2.2 Feynman diagrams of the s and t-channel of electron-positron interaction.

The denominators coming from the perturbation expansion cancel, because the matrix elements double for one and the same process. Therefore, omitting the spinor and chirality indices the matrix elements can be written as follows:

$$M_s = ie^2 \bar{u}(p_1')\gamma^{\mu} v(p_2') \frac{1}{(p_1 + p_2)^2} \bar{v}(p_2)\gamma_{\mu} u(p_1) \tag{2.23}$$

$$M_t = -ie^2 \bar{u}(p_1')\gamma_{\mu} u(p_1) \frac{1}{(p_1 - p_1')^2} \bar{v}(p_2)\gamma^{\mu} v(p_2') \tag{2.24}$$

Using trace techniques of Dirac matrices and summing over spinor indices we find the square of the total matrix element.

$$\sum |\bar{M}|^2 = M^{\dagger} M = M_s^2 + M_t^2 - 2\mathbb{R}(M_s^{\dagger} M_t) \tag{2.25}$$

For the sake of simplicity we work in the ultra-relativistic limit and we get

$$\begin{aligned} M_s^2 &= \frac{e^4}{s^2} \bar{v}(p_2')\gamma^{\nu} u(p_q')\bar{u}(p_1)\gamma_{\nu} v(p_2)\bar{u}(p_1')\gamma^{\mu} v(p_2')\bar{v}(p_2)\gamma_{\mu} u(p_1) = \\ &= \frac{e^4}{s^2} \mathrm{Tr}\left[\gamma^{\nu}(\hat{p}_1' + m)\gamma^{\mu}(\hat{p}_2' - m)\right] \mathrm{Tr}\left[\gamma_{\nu}(\hat{p}_2 - m)\gamma_{\mu}(\hat{p}_1 + m)\right] \to \\ &\to \frac{32e^4}{s^2}\left[(p_1 p_2')(p_1' p_2) + (p_1 p_1')(p_2 p_2')\right] \end{aligned} \tag{2.26}$$

Analogously,

$$M_t^2 = \frac{32e^4}{t^2}\left[(p_1'p_2)(p_1p_2') + (p_1p_2)(p_1'p_2') - m^2(p_1p_1') - m^2(p_2p_2') + 2m^4\right] \to \\ \to \frac{32e^4}{t^2}\left[(p_1'p_2)(p_1p_2') + (p_1p_2)(p_1'p_2')\right] \tag{2.27}$$

And the cross term gives

$$M_{st} = -\frac{e^4}{st}\bar{v}(p_2')\gamma_\nu u(p_1')\bar{u}(p_1)\gamma^\nu v(p_2)\bar{u}(p_1')\gamma_\mu u(p_1)\bar{v}(p_2)\gamma^\mu v(p_2') = \\ = \frac{e^4}{st}\mathrm{Tr}\left[(\hat{p}_1 + m)\gamma_\nu(\hat{p}_2 - m)\gamma_\mu(\hat{p}_2' - m)\gamma^\nu(\hat{p}_1' + m)\gamma^\mu\right] \to \\ \to \frac{32e^4}{st}(p_1p_2')p_1'p_2) \tag{2.28}$$

Due to summing over spins we need to add a factor of a quarter, therefore

$$M = 8e^4\left[\frac{(p_1p_2')(p_1'p_2)}{s^2} + \\ + \frac{(p_1p_1')(p_2p_2')}{s^2}\frac{(p_1'p_2)(p_1p_2') + (p_1p_2)(p_1'p_2')}{t^2} + \frac{2(p_1p_2')(p_1'p_2)}{st}\right] \tag{2.29}$$

Writing in terms of the Mandelstam variables

$$s = (p_1 + p_2)^2 = 2p_1p_2 = 2p_1'p_2' \tag{2.30}$$

$$t = (p_1 - p_1')^2 = -2p_1p_1' = -2p_2p_2' \tag{2.31}$$

$$u = (p_2' - p_1)^2 = -2p_1p_2' = -2p_1'p_2 \tag{2.32}$$

the final result is

$$M^2 = 2e^4\left[u^2\left(\frac{1}{s} + \frac{1}{t}\right)^2 + \frac{t^2}{s^2} + \frac{s^2}{t^2}\right] \tag{2.33}$$

The unpolarized cross-section in the centre of mass frame is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}M^2 = \frac{\alpha^2}{2s}\left[u^2\left(\frac{1}{s} + \frac{1}{t}\right)^2 + \frac{t^2}{s^2} + \frac{s^2}{t^2}\right] \tag{2.34}$$

2.2 Discrete Symmetries

In this section we will consider the existing discrete symmetries that spinor fields can obey. Later we will also discuss quantum field theories with certain continuous symmetries and the mechanism of spontaneous symmetry breaking giving rise to the mass of particles. The concept of symmetry is pivotal in this work. One of our main objectives is by starting from first principles to be able to predict the existence of isospin symmetry and its conservation. In 1932 isospin symmetry was introduced by W. Heisenberg in order to explain the equivalence in mass between the proton and the neutron and we will later show how the Nambu-Jona-Lasinio model might be a possible explanation for the existence of isospin .

We start with the equation governing the dynamics of a massive fermion - the famous Dirac equation

$$(i\hat{\partial} - m)\Psi = 0 \tag{2.35}$$

and every solution of the Dirac equation can be represented by a combination of two component objects called Weyl spinors

$$\Psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \tag{2.36}$$

where we have adopted the Weyl representation of the gamma matrices in which

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{2.37}$$

$i = 1,2,3$ and σ^i are the Pauli matrices. In this representation we can define chiral (left or right) spinors

$$\Psi_\pm = P_\pm \Psi, \text{ with } P_\pm = \frac{1}{2}(1 \pm \gamma^5) \tag{2.38}$$

being a projection operator. There exist a further connection between left and right spinors which is called parity. It is a type of a discrete symmetry such that it reverses the space components of the spinor:

$$P : \Psi_\pm(\vec{x},t) \to \Psi_\mp(-\vec{x},t) \tag{2.39}$$

From the definition of P we can note that $P^2 = 1$ and a suitable matrix representation for P is $P = \gamma^0$.

For charged particles we can also define another type of discrete symmetry - charge symmetry, or C-symmetry. It is a transformation that turns a particle into its anti-partner and is well defined only for particles with all conserved charges equal to 0. Written mathematically,

$$C\Psi = \bar{\Psi} \tag{2.40}$$

The matrix C obeys the properties

$$C = C^{\dagger} \text{ and } C^{\dagger}C = 1 \tag{2.41}$$

The final symmetry we discuss is the time reversal symmetry, or T symmetry, which changes the sign of the time component of a spinor. In electroweak theory it has been observed that C and P symmetries are simultaneously violated - CP violation. This phenomenon was observed in the decay of neutral kaons in 1964 which earned a Nobel prize for James Cronin and Val Fitch in 1980. This observation has also a strong and complicated consequences in cosmology related to the dominance of matter over anti-matter. But for any physical, local and Lorentz invariant theory the combination of the three discrete symmetries must be conserved, and that is called the CPT theorem. Finally, there exist one more quantum number, called G-parity, which is a combination between C symmetry and isospin rotation:

$$G = Ce^{-i\pi I_2} \tag{2.42}$$

Later in this thesis important simplifications of the presented model will be made on symmetry grounds.

2.3 Neutral Pion Decay into Photons

One important problem which will be discussed in further chapters of the thesis is Feynman loop diagrams with three external lines, where one is a pseudoscalar and the other two are vector particles. We will construct a Lagrangian containing both pseudoscalar and vector sector and will calculate all possible loop diagram. For this reason we will step on a similar and well-known process, namely the decay of a neutral pion into two photons. This process has a very important consequence - the Goldberger-Treiman relation which follows from the nonconservation of axial current (PCAC).

We construct a local Lorentz invariant Lagrangian describing electromagnetic interaction between the quarks and effective strong interaction without including derivatives of fields.

$$\mathscr{L} = \mathscr{L}_{\text{strong}} + \mathscr{L}_{\text{EM}} = ig\bar{\Psi}\gamma^5\vec{\tau}\Psi\vec{\pi} - e\left(\frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d\right)A^\mu, \tag{2.43}$$

$$\Psi = \begin{pmatrix} u \\ d \end{pmatrix} \qquad \vec{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} \tag{2.44}$$

where $\vec{\tau}$ are the Pauli matrices and $\pi_3 = \pi^0$. Introducing $\Gamma^5 = \gamma^5\tau^3$ and $C = \frac{1}{6}\tau^0 + \frac{1}{2}\tau^3$ the interaction Lagrangian for the neutral pion becomes

$$\mathscr{L} = ig\bar{\Psi}(x)\Gamma^5\Psi(x)\pi^0(x) - e\bar{\Psi}(x)C\hat{A}(x)\Psi(x) \tag{2.45}$$

Let the scattering matrix be

$$S = \frac{i^3}{3!}\int dxdydzS_3(x,y,z), \tag{2.46}$$

where

$$\begin{aligned} S_3(x,y,z) = T\left[\mathscr{L}(x)\mathscr{L}(y)\mathscr{L}(z)\right] = T\Big[\Big(&:ig\bar{\Psi}(x)\Gamma^5\Psi(x)\pi^0(x): - e:\bar{\Psi}(x)C\hat{A}(x)\Psi(x):\Big) \\ \Big(&:ig\bar{\Psi}(y)\Gamma^5\Psi(y)\pi^0(y): - e:\bar{\Psi}(y)C\hat{A}(y)\Psi(y):\Big) \\ \Big(&:ig\bar{\Psi}(z)\Gamma^5\Psi(z)\pi^0(z): - e:\bar{\Psi}(z)C\hat{A}(z)\Psi(z):\Big)\Big] \end{aligned} \tag{2.47}$$

We are only interested in diagrams, which involve triangle fermionic loops. The scattering matrix will be the sum of six different terms and should be noted that only the first two are written explicitly. The next two are obtained by first changing x into y and the last two by changing x into z, y into x and z into y.

$$\begin{aligned} S_{31}(x,y,z) = ige^2\mathrm{T}\Big[&:\bar{\Psi}(x)\Gamma^5\Psi(x)\pi^0(x)::\bar{\Psi}(y)C\hat{A}(y)\Psi(y)::\bar{\Psi}(z)C\hat{A}(z)\Psi(z):+ \\ &+:\bar{\Psi}(x)\Gamma^5\Psi(x)\pi^0(x)::\bar{\Psi}(y)C\hat{A}(y)\Psi(y)::\bar{\Psi}(z)C\hat{A}(z)\Psi(z):+ \\ &+(x\leftrightarrow y)+(x\rightarrow z\ \ y\rightarrow x\ \ z\rightarrow y)\Big] \end{aligned} \tag{2.48}$$

Using the equation below, and also adopting the approximation that the constituent mass of the up and down quark is approximately the same $m_u \approx m_d \approx \frac{1}{3}m_p \equiv m$ (the constituent mass of the up quark is 336 MeV/c^2).

$$\overbracket{\Psi_\alpha(x)\bar{\Psi}}(y)_\beta = -iS^c_{\alpha\beta}(x-y) = -\frac{i}{(2\pi)^4}\int e^{-ip(x-y)}\frac{(m+\hat{p})_{\alpha\beta}}{m^2-p^2+i\varepsilon}dp \tag{2.49}$$

we obtain

$$\begin{aligned} S_{31}(x,y,z) = {} & ge^2\mathrm{Tr}\left\{\Gamma^5 S^c(x-y)C\hat{A}(y)S^c(y-z)C\hat{A}(z)S^c(z-x)\right\} : \pi^0(x) : + \\ & + ge^2\mathrm{Tr}\left\{\Gamma^5 S^c(x-z)C\hat{A}(z)S^c(z-y)C\hat{A}(y)S^c(y-x)\right\} : \pi^0(x) : + \\ & + ge^2\mathrm{Tr}\left\{\Gamma^5 S^c(y-x)C\hat{A}(x)S^c(x-z)C\hat{A}(z)S^c(z-y)\right\} : \pi^0(y) : + \\ & + ge^2\mathrm{Tr}\left\{\Gamma^5 S^c(y-z)C\hat{A}(z)S^c(z-x)C\hat{A}(x)S^c(x-y)\right\} : \pi^0(y) : + \\ & + ge^2\mathrm{Tr}\left\{\Gamma^5 S^c(z-x)C\hat{A}(x)S^c(x-y)C\hat{A}(y)S^c(y-z)\right\} : \pi^0(z) : + \\ & + ge^2\mathrm{Tr}\left\{\Gamma^5 S^c(z-y)C\hat{A}(y)S^c(y-x)C\hat{A}(x)S^c(x-z)\right\} : \pi^0(z) : \end{aligned} \tag{2.50}$$

The scattering matrix we are interested in reduces to

$$\begin{aligned} S'_3 = \frac{ge^2}{2}\int dxdydz : \pi^0(x) : \Big\{ & \mathrm{Tr}\left[\Gamma^5 S^c(x-y)C\hat{A}(y)S^c(y-z)C\hat{A}(z)S^c(z-x)\right] \\ & + \mathrm{Tr}\left[\Gamma^5 S^c(x-z)C\hat{A}(z)S^c(z-y)C\hat{A}(y)S^c(y-x)\right]\Big\} \end{aligned} \tag{2.51}$$

The corresponding Feynman diagrams are the following:

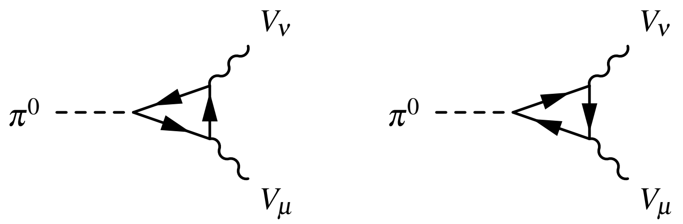


Fig. 2.3 The diagrams corresponding to the terms in 2.51

In this case the *in* state is the creation of a pion from vacuum, and the *out* state is the Hermitian conjugate of the creation of two photons. $|\text{in}\rangle = a^+(p)\Phi_0$ and $\langle\text{out}| = \Phi_0^\dagger a^-_{\nu_2}(k_2)a^-_{\nu_1}(k_1)$ and the matrix element can be obtained as the sandwich of the scattering matrix between the in and out states. We should further define the Fourier decomposition of the particle states:

$$\pi^0(x) = \int \frac{d\vec{p}}{\sqrt{2p^0}} \left(a^+ \vec{p} e^{ipx} + a^- \vec{p} e^{-ipx} \right) \tag{2.52}$$

$$A_\mu(x) = \int dk \delta(k^2) \left(a^+_\nu(k) \varepsilon^\nu_\mu(k) e^{ikx} + a^-_\nu(k) \varepsilon^\nu_\mu(k) e^{-ikx} \right) \tag{2.53}$$

Then we construct the matrix element by contracting the non-vanishing combinations between the operators in the scattering matrix and the operators of creation and annihilation in the in and out states.

$$\begin{aligned} M = \langle \text{out}| \; S'_3 \; |\text{in}\rangle = n_c \frac{ge^2}{2} \text{Tr}[\tau^3 CC] \Phi_0^\dagger a^-_{\nu_2}(k_2) a^-_{\nu_1}(k_1) \\ \int dxdydz : \pi^0(x) : \left\{ \text{Tr} \left[\gamma^5 S^c(x-y) \hat{A}(y) S^c(y-z) \hat{A}(z) S^c(z-x) \right] + \right. \\ \left. + \text{Tr} \left[\gamma^5 S^c(x-z) \hat{A}(z) S^c(z-y) \hat{A}(y) S^c(y-x) \right] \right\} a^+(p) \Phi_0 \end{aligned} \tag{2.54}$$

The reason for which we have two different Feynman diagrams is that the photons are indistinguishable particles and consequently there are two analogous ways of defining the direction of the momenta inside the loop. The factor $n_c = 3$ is a colour factor which cancels $\text{Tr}[\tau_3 CC] = \frac{1}{3}$. We define l to be the incoming momentum from the loop towards the pion vertex. Below are performed the possible contractions for the first term of the scattering matrix. The contractions of the second term are done in analogous way. Let

$$\begin{aligned} M_1 = -\frac{ge^2}{2(2\pi)^{12}} \Phi_0^\dagger \int dxdydz \; \text{Tr} \Big\{ \\ a^-_{\nu_2}(k_2) a^-_{\nu_1}(k_1) \pi^0(x) \gamma^5 S^c(x-y) \gamma^\mu A_\mu(y) S^c(y-z) \gamma^\nu A_\nu(z) S^c(z-x) a^+(p) - \\ - a^-_{\nu_2}(k_2) a^-_{\nu_1}(k_1) \pi^0(x) \gamma^5 S^c(x-y) \gamma^\mu A_\mu(y) S^c(y-z) \gamma^\nu A_\nu(z) S^c(z-x) a^+(p) \Big\} \Phi_0 \end{aligned} \tag{2.55}$$

Applying the commutation relations for vector and scalar fields, the non-vanishing terms are:

$$\Phi_0^\dagger a^-_{\nu_2}(k_2) A_\nu(z) = \int dk'_2 \delta(k_2 - k'_2) \delta_{\nu_2 \nu_{2'}} \varepsilon_\nu^{\nu_{2'}}(k_{2'}) e^{ik'_2 z} = \varepsilon_\nu^{\nu_2}(k_2) e^{ik_2 z} \tag{2.56}$$

$$\Phi_0^{\dagger} a^{-}_{\nu_1}(k_1) A_\mu(y) = \int dk'_1 \delta(k_1 - k'_1)\delta_{\nu_1 \nu_{1'}} \varepsilon_\mu^{\nu_{1'}}(k_{1'}) e^{ik'_1 y} = \varepsilon_\mu^{\nu_1}(k_1) e^{ik_1 y} \tag{2.57}$$

$$\pi^0(x) a^{+}(p)\Phi_0 = \int dp' \delta(p - p') e^{-ip'x} = e^{-ipx} \tag{2.58}$$

The first and the second part of the matrix element M_1 are equal, since the two photon contractions lead to equivalent processes. Therefore, substituting all of the simplified equations we obtained for M_1 we get

$$M_1 = -\frac{ge^2}{(2\pi)^{12}} \int dxdydz e^{-ipx} e^{-il(x-y)} dl e^{-ip_1(z-x)} dp_1 e^{-ip_2(y-z)} dp_2 e^{ik_1 y} e^{ik_2 z} \varepsilon_\mu^{\nu_1}(k_1) \varepsilon_\nu^{\nu_2}(k_2) \\ \frac{\mathrm{Tr}\left[\gamma^5(\hat{l}+m)\gamma_\mu(\hat{l}+\hat{k}_1+m)\gamma_\nu(\hat{l}+\hat{p}+m)\right]}{(l^2-m^2)[(l+k_1)^2-m^2][(l+p)^2-m^2]} \tag{2.59}$$

In exactly the same way we obtain that

$$M_2 = -\frac{ge^2}{(2\pi)^{12}} \int dxdydz e^{-ipx} e^{-il(x-y)} dl e^{-ip_1(z-x)} dp_1 e^{-ip_2(y-z)} dp_2 e^{ik_1 y} e^{ik_2 z} \varepsilon_\mu^{\nu_1}(k_1) \varepsilon_\nu^{\nu_2}(k_2) \\ \frac{\mathrm{Tr}\left[\gamma^5(\hat{l}-\hat{p}+m)\gamma_\nu(\hat{l}-\hat{k}_1+m)\gamma_\mu(\hat{l}+m)\right]}{(l^2-m^2)[(l-k_1)^2-m^2][(l-p)^2-m^2]} \tag{2.60}$$

The integration over the momenta inside the loop results in a single Dirac delta function and therefore - conservation of 4-momentum at each vertex. First, we perform the integration over the coordinates x, y and z, left from the scattering matrix equation, and then we integrate over momenta.

$$\int dx \exp[-i(p+l-p_1)x] = (2\pi)^4 \delta^{(4)}(p+l-p_1) \tag{2.61}$$

$$\int dy \exp[-i(-l+p_2-k_1)y] = (2\pi)^4 \delta^{(4)}(-l+p_2-k_1) \tag{2.62}$$

$$\int dz \exp[-i(p_1-p_2-k_1)z] = (2\pi)^4 \delta^{(4)}(p_1-p_2-k_1) \tag{2.63}$$

Substituting these results back gives

$$\int dl dp_1 dp_2 \delta^{(4)}(p+l-p_1)\delta^{(4)}(-l+p_2-k_1)\delta^{(4)}(p_1-p_2-k_1) = (2\pi)^4 \int \frac{dl}{(2\pi)^4}\delta^{(4)}(p-k_1-k_2) \tag{2.64}$$

where k_1 and k_2 are the outgoing momenta of the photons, and p is the pion momentum. By definition

$$S_{if} = 1 + T_{if} = (2\pi)^4\delta^{(4)}(\Sigma_i p_i)M \Rightarrow \tag{2.65}$$

$$\begin{aligned} M = -ige^2\varepsilon_\mu^{\nu_1}(k_1)\varepsilon_\nu^{\nu_2}(k_2)\int \frac{d^4l}{(2\pi)^4}\Bigg\{ & \\ \frac{\mathrm{Tr}\left[\gamma^5(\hat{l}+m)\gamma_\mu(\hat{l}+\hat{k}_1+m)\gamma_\nu(\hat{l}+\hat{p}+m)\right]}{(l^2-m^2+i\varepsilon)[(l+k_1)^2-m^2+i\varepsilon][(l+p)^2-m^2+i\varepsilon]} + & \\ + \frac{\mathrm{Tr}\left[\gamma^5(\hat{l}-\hat{p}+m)\gamma_\nu(\hat{l}-\hat{k}_1+m)\gamma_\mu(\hat{l}+m)\right]}{(l^2-m^2+i\varepsilon)[(l-k_1)^2-m^2+i\varepsilon][(l-p)^2-m^2+i\varepsilon]}\Bigg\} & \end{aligned} \tag{2.66}$$

The trace parts of the matrix element gives

$$\mathrm{Tr}\Big[\gamma^5(\hat{l}+m)\gamma^\mu(\hat{l}+\hat{k}_1+m)\gamma^\nu(\hat{l}+\hat{p}+m)\Big] = -4im\varepsilon^{pk_1\mu\nu} \tag{2.67}$$

$$\mathrm{Tr}\Big[\gamma^5(\hat{l}-\hat{p}+m)\gamma^\nu(\hat{l}-\hat{k}_1+m)\gamma^\mu(\hat{l}+m)\Big] = -4im\varepsilon^{pk_1\mu\nu} \tag{2.68}$$

given that the convention we adopt is that $\mathrm{Tr}(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu) = -4i\varepsilon^{\alpha\beta\mu\nu}$. It can be shown that the matrix element for the second part of the scattering matrix is the same as the first when switching $l \to -l$. Finally, the total matrix element is

$$\begin{aligned} M = -8ge^2m\varepsilon_\mu^{\nu_1}(k_1)\varepsilon_\nu^{\nu_2}(k_2)\varepsilon^{pk_1\mu\nu} & \\ \int \frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-m^2+i\varepsilon)[(l+k_1)^2-m^2+i\varepsilon][(l+p)^2-m^2+i\varepsilon]} & \end{aligned} \tag{2.69}$$

To find the integral involved we use Feynman parameters method. Let

$$I = \int \frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-m^2+i\varepsilon)[(l+k_1)^2-m^2+i\varepsilon][(l+p)^2-m^2+i\varepsilon]} \tag{2.70}$$

We make use of the formula

$$\prod_{i=1}^{n}\frac{1}{A_i}=\int_0^1\prod_{i=1}^{n}da_i\delta\left(\sum_i a_i-1\right)\frac{(n-1)!}{(\sum_i a_iA_i)^n} \tag{2.71}$$

Therefore, our integral turns into

$$\int\frac{d^4l}{(2\pi)^4}\int_0^1 dadbdc\ \ \delta(a+b+c-1)\frac{2}{D^3}, \tag{2.72}$$

where $D=a\cdot l^2+b(l+k_1)^2+c(l+p)^2-m^2+i\varepsilon=l^2+2l\cdot(bk_1+cp)+c\cdot p^2-m^2+i\varepsilon$ Introduce $q=l+b\cdot k_1+c\cdot p$ to complete the square and we obtain for the denominator

$$D=q^2-2bc(k_1\cdot p)-c^2\cdot p^2+c\cdot p^2-m^2+i\varepsilon=q^2-\Delta+i\varepsilon \tag{2.73}$$

where $\Delta=m_\pi^2(bc+c^2-c)+m^2$ and m_π is the pion mass and m is the quark mass. Here we have used the conservation of momentum $p=k_1+k_2$ and therefore, using that photons are massless $p^2=m_\pi^2=2(k_1k_2)$

The integral becomes

$$I=\int\frac{d^4q}{(2\pi)^4}\int_0^1 dadbdc\frac{2\delta(a+b+c-1)}{((q^2-\Delta+i\varepsilon)^3} \tag{2.74}$$

Now we perform a Wick rotation by which we make the integral from Minkowski space-time into 4D Euclidean space. This is done by applying a specific change of variables $q^0=iq_E^0$, $\vec{q}=\vec{q}_E$. Also,

$$\int\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2-\Delta)^m}=\frac{i(-1)^m}{(2\pi)^4}\int d\Omega_4\int_0^\infty dq_E\frac{q_E^3}{(q_E^2+\Delta)^m} \tag{2.75}$$

where $\int d\Omega_4$ is the surface area of a 4D unit sphere, which is equal to $2\pi^2$. Then,

$$\int_0^\infty dq_E\frac{q_E^3}{(q_E^2+\Delta)^3}=\frac{1}{4\Delta} \tag{2.76}$$

and therefore

$$I=\int dadbdc\frac{-i}{16\pi^2}\frac{\delta(a+b+c-1)}{\Delta}=-\frac{i}{16\pi^2}\int dadbdc\frac{\delta(a+b+c-1)}{m_\pi^2(bc+c^2-c)+m^2}$$

$$= -\frac{i}{16\pi^2}\int_{c=0}^{1} dc \int_{b=0}^{1-c} db \frac{1}{m_\pi^2(bc+c^2-c)+m^2} =$$

$$= -\frac{i}{16\pi^2 m_\pi^2}\left[\text{PolyLog}\left(2, \frac{2m_\pi}{m_\pi - \sqrt{m_\pi^2 - 4m^2}}\right) + \text{PolyLog}\left(2, \frac{2m_\pi}{m_\pi + \sqrt{m_\pi^2 - 4m^2}}\right)\right]$$

$$\to -\frac{i}{32\pi^2 m^2} \text{ for } m_\pi \to 0 \quad (2.77)$$

The resultant matrix element in the case of a massless pion

$$M = i\frac{ge^2}{4\pi^2 m}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_\mu^{\nu_1}(k_1)\varepsilon_\nu^{\nu_2}(k_2)k_{1\alpha}k_{2\beta} \quad (2.78)$$

The Goldberger-Treiman relation gives us the ratio between the constant g and the constituent quark mass m:

$$g = \frac{mg_A}{f_\pi}, \text{ where we take } f_\pi = 93MeV/c^2 \text{ and in our case } g_A = 1 \quad (2.79)$$

This helps us to write the matrix element in the simplest possible form:

$$M = i\frac{e^2}{4\pi^2 f_\pi}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_\mu^{\nu_1}(k_1)\varepsilon_\nu^{\nu_2}(k_2)k_{1\alpha}k_{2\beta} \quad (2.80)$$

Working in natural units $c = \hbar = 1$ then $e = 0.303$, $m = 336\ MeV/c^2$, $m_\pi = 135\ MeV/c^2$. In the special case when there is a decay of a single particle into two particles, a simplified formula for the decay rate could be applied

$$\Gamma = \frac{1}{2m_\pi}\frac{1}{8\pi}\frac{1}{2}|M|^2 = \frac{1}{32\pi m_\pi}\left(\frac{e^2}{4\pi^2 f_\pi}\right)^2 2(k_1\cdot k_2)^2 = \frac{e^4 m_\pi^3}{1024\pi^5 f_\pi^2} = 7.652eV \quad (2.81)$$

where an additional factor of $\frac{1}{2}$ is added because of the indistinguishability of the two photons. Substituting all known values we obtain that the lifetime of the neutral pion is about $\tau = 8.576 \times 10^{-17}$s.

2.4 Renormalization

We continue with the discussion of another technique which will be used throughout the whole thesis - renormalization. In quantum field theory the purely quantum effects like self interaction of particles gives rise to infinities in the Lagrangian. People realized that the masses and the charges of particles that are present in the equations of motion are not the quantities we measure in experiments. The need for renormalization becomes obvious when we take into account interactions of a particle with virtual particles. In order for the theory to be physical and lacking all the arising divergences we introduce the so-called counter-terms in the Lagrangian which cancel out the infinite terms. That is why we have *bare* and *renormalized* quantities. Many people have been working on renormalization and the foundations include the works of N. Bogoliubov [31] on proving that for a renormalized theory the matrix element and the Green's function are finite (Bogoliubov-Parasyuk theorem), then F. Dyson showed that QED theory invented by R. Feynman, S. Tomonaga and J. Schwinger can be renormalized and the divergences in the scattering matrix can be removed at all orders [32]. A big problem in modern theoretical physics is that when we try to create a quantum theory of gravity and renormalize it there exist infinitely many divergencies at all orders, so quantum theory of gravity is a non-renormalizable theory. The next step was to show that non-Abelian theories can be renormalized. G. t'Hooft showed that general Yang-Mills theories (both with massless and massive fields) are renormalizable [33, 34], and also F. Wilzcek, D. Gross and H. Politzer showed that quantum chromodynamics (QCD) is asymptotically free [12, 35]. This property leads to the consequence that for small energies QCD in not perturbative and different models need to be applied to account for strong interactions at these energy scale.

To illustrate the process of renormalization we will consider QED to 1 loop as it offers examples closest to what will be done in further chapters of the thesis. We start with the following Lagrangian:

$$\mathscr{L}_{\text{QED}} = i\bar{\psi}\gamma^{\mu}\psi - m\bar{\psi}\psi - eA_{\mu}\bar{\psi}\gamma^{\mu}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.82}$$

There exist three strongly connected graphs which we will consider

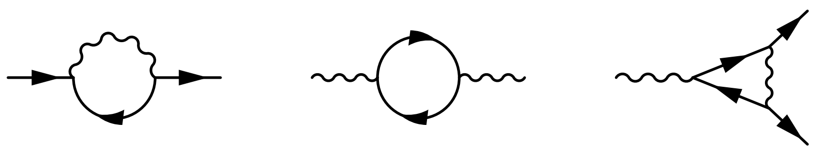


Fig. 2.4 The three non-zero divergent diagrams up to 1 loop in QED.

In order to see that these are actually divergent diagram, the general formula for the index of a diagram is the following:

$$D = nL - 2P_i - E_i, \tag{2.83}$$

where D is the index of the diagram, n is the number of space-time dimensions, which in our case is 4, L is the number of loops, P_i is the number of internal photon lines, and E_i is the number of internal electron lines. Therefore, the correction to the fermion self-energy is linearly divergent, the vacuum polarization diagram is quadratically divergent, and the correction to the photon-fermion vertex is logarithmically divergent. In addition there could be a fermion loop with three external photons, which does not give a contribution due to Furry's theorem (following from C-invariance of the theory), photon-photon scattering, which at a first sight should be logarithmically divergent, but turns out to be convergent due to gauge invariance of QED.

For the first diagram, following the Feynman rules we get

$$i\Sigma(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(\hat{p}-\hat{k}+m)}{(p-k)^2 - m^2 + i\varepsilon} \frac{-ig_{\mu\nu}}{k^2+i\varepsilon} \gamma^\nu \tag{2.84}$$

We will apply dimensional regularisation and the effective Lagrangian becomes

$$-ie^2\mu^{4-d} \int \frac{d^dk}{(2\pi)^d} \gamma^\mu \frac{\hat{p}-\hat{k}+m}{[(p-k)^2 - m^2 + i\varepsilon]\,(k^2+i\varepsilon)} \gamma_\mu \tag{2.85}$$

Throughout the whole document we will use the following generalised Dirac matrix identities in d-dimensions:

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta] = 4(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}) \tag{2.86}$$

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu] = 4g_{\mu\nu} \tag{2.87}$$

$$\mathrm{Tr}(I) = 4 \tag{2.88}$$

$$\gamma_\mu\gamma_\nu\gamma^\mu = (2-d)\gamma_\nu \tag{2.89}$$

$$\gamma^\mu\gamma_\mu = d \tag{2.90}$$

Then the Lagrangian in d-dimensions becomes

$$\Sigma(p) = -ie^2\mu^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{(2-d)(\hat{p}-\hat{k}) + dm}{[(p-k)^2 - m^2 + i\varepsilon](k^2+i\varepsilon)} \tag{2.91}$$

In order to calculate the above integral we make use of the following property

$$\prod_{i=1}^{n}\frac{1}{A_i}=\int_0^1\prod_{i=1}^{n}da_i\delta\left(\sum_i a_i-1\right)\frac{(n-1)!}{(\sum_i a_iA_i)^n} \tag{2.92}$$

Applying this formula to $\Sigma(p)$ we get that the denominator D becomes

$$D=\left[x(p-k)^2-xm^2+i\varepsilon+yk^2\right]^2=\left[xp^2-2x(pk)+k^2-xm^2+i\varepsilon\right]^2, \tag{2.93}$$

where x is a Feynman parameter. Now we make a change of variables such that $k'=k-xp$. Then $k'^2=k^2-2x(pk)+x^2p^2$ and the Lagrangian becomes

$$-ie^2\int\frac{d^dk'}{(2\pi)^d}\int_0^1 dx\frac{(2-d)(\hat{p}-\hat{k}'-x\hat{p})+dm}{[k'^2+xp^2(1-x)-xm^2+i\varepsilon]^2} \tag{2.94}$$

The linear terms in k' integrate to 0 and we get

$$\int\frac{d^dk'}{(2\pi)^d}\int_0^1 dx\frac{(2-d)(1-x)\hat{p}+dm}{[k'^2-x(1-x)p^2-xm^2]^2} \tag{2.95}$$

Throughout the whole document we will apply two integrals, valid only in Minkowski space-time

$$\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2-\Delta)^n}=\frac{(-1)^n i}{(4\pi)^{d/2}}\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \tag{2.96}$$

$$\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{(l^2-\Delta)^n}=\frac{(-1)^{n-1}i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma\left(n-\frac{d}{2}-1\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{2.97}$$

$$\Sigma(p)=\int_0^1 dx(-1)^{d-2}i\pi^{d/2}\Gamma\left(2-\frac{d}{2}\right)\frac{(2-d)(1-x)\hat{p}+dm}{[x(1-x)p^2+xm^2]^{2-\frac{d}{2}}} \tag{2.98}$$

We will express the number of dimensions as $d=4-2\varepsilon$ and will further take ε to be tending to 0. Retaining only divergent terms we finally obtain that

$$\Sigma(p)=\frac{e^2(4m-\hat{p})}{8\pi^2\varepsilon}+\text{finite terms},\varepsilon\to 0 \tag{2.99}$$

For the vacuum polarization diagram, following the Feynman rules the effective Lagrangian is

$$\Pi_{\mu\nu}(q^2)=ie^2\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[\gamma_\mu\frac{\hat{p}+m}{p^2-m^2}\gamma_\nu\frac{\hat{p}-\hat{k}+m}{(p-k)^2-m^2}\right] \tag{2.100}$$

Written in d-dimensions this effective Lagrangian becomes

$$\Pi_{\mu\nu}(q^2) = ie^2\mu^{4-d}\int \frac{d^d p}{(2\pi)^d}\mathrm{Tr}\left[\gamma_\mu \frac{\hat{p}+m}{p^2-m^2}\gamma_\nu \frac{\hat{p}-\hat{k}+m}{(p-k)^2-m^2}\right] \tag{2.101}$$

The denominator in the above equation we denote by $D = (p^2-m^2)[(p-k)^2-m^2]$. After the application of the Feynman parameters technique, we can write the whole denominator expression in terms of one parameter y and a redefined momentum $l = p - yk$:

$$D = (l^2-\Delta)^2, \tag{2.102}$$

where $\Delta = y(y-1)k^2+m^2$. Using the trace identities for a general number of space-time dimensions we can write the numerator in terms of l as

$$N_{\mu\nu} = 4\left[-\frac{1}{2}g_{\mu\nu}l^2 + 2y(y-1)k_\mu k_\nu - y(y-1)k^2 g_{\mu\nu} + m^2 g_{\mu\nu}\right] \tag{2.103}$$

and the Lagrangian will be

$$\begin{aligned}\Pi_{\mu\nu}(q^2) = -4ie^2\mu^{4-d}\int \frac{d^d l}{(2\pi)^d}\int_0^1 dy \\ \frac{\frac{1}{2}g_{\mu\nu}l^2 - 2y(y-1)k_\mu k_\nu + y(y-1)k^2 g_{\mu\nu} - m^2 g_{\mu\nu}}{(l^2-\Delta)^2}\end{aligned} \tag{2.104}$$

Perform a Wick rotation such that $l^0 \to il^o_E$ and $\vec{l} \to \vec{l}_E$ and the whole integral becomes

$$\begin{aligned}\Pi_{\mu\nu}(q^2) = -4ie^2\mu^{4-d}\int_0^\infty \frac{d^d l_E}{(2\pi)^d}\int_0^1 dy \\ \frac{\frac{1}{2}g_{\mu\nu}l_E^2 - 2y(1-y)k_\mu k_\nu + g_{\mu\nu}(m^2+y(1-y)l_E^2)}{\left[l_E^2 - y(1-y)l_E^2+m^2\right]^2}\end{aligned} \tag{2.105}$$

We have essentially two integrals - one with a numerator proportional to l^2 and one with a numerator without any l involved.

$$\int_0^\infty \frac{d^d l_E}{(2\pi)^d} - \frac{1}{2}g_{\mu\nu}\frac{l_E^2}{(l_E^2+\Delta)^2} = \frac{1}{2}g_{\mu\nu}\frac{i}{(4\pi)^{d/2}}\frac{d}{2}\Gamma\left(1-\frac{d}{2}\right)\frac{1}{(-\Delta)^{1-\frac{d}{2}}} \tag{2.106}$$

In the limit when $d \to 4$ this integral becomes $\frac{ig_{\mu\nu}\Delta}{8\pi^2\varepsilon}$. The integral not involving l in the numerator is

$$\int_0^\infty \frac{d^d l_E}{(2\pi)^d} \frac{[...]}{(l_E^2-\Delta)^2} = [...] \frac{i}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) \frac{1}{(-\Delta)^{2-d/2}} \to [...] \frac{i}{8\pi^2\varepsilon} \tag{2.107}$$

Having these results we can finalize the Lagrangian computation by performing the integration over y

$$\begin{aligned}\Pi_{\mu\nu} = \frac{ie^2}{2\pi^2\varepsilon}\int_0^1 dy\left[y(y-1)k^2 g_{\mu\nu} + 2y(1-y)k_\mu k_\nu - y(1-y)k^2 g_{\mu\nu}\right] = \\ = \frac{ie^2}{6\pi^2\varepsilon}(k_\mu k_\nu - k^2 g_{\mu\nu})\end{aligned} \tag{2.108}$$

In the vertex correction diagram we get the following effective Lagrangian, written in d-dimensions

$$\begin{aligned}\Gamma^\mu(p,p') =& \mu^{4-d}\int \frac{d^d k}{(2\pi)^d} \\ & (-ie\gamma_\nu) i\frac{\hat{k}+\hat{p}+m}{(k+p)^2-m^2+i\varepsilon}\gamma^\mu i\frac{\hat{k}+\hat{p}'+m}{(k+p')^2-m^2+i\varepsilon}(-ie\gamma^\nu)\frac{-i}{k^2+i\varepsilon}\end{aligned} \tag{2.109}$$

Introduce Feynman parameters, where we use the kinematics of the particles (the on-shell photon has $k^2=0$ and the off-shell photon $q^2 \neq 0$)

$$\begin{aligned}D = \left\{x\left[(k+p)^2-m^2+i\varepsilon\right] + y\left[(k+p')^2-m^2+i\varepsilon\right] + z(k^2+i\varepsilon)\right\}^3 = \\ = \left[k^2+2k(xp+yp')+i\varepsilon\right]^3\end{aligned} \tag{2.110}$$

Now perform a change in variables $l = k + xp + yp'$. Also, following from the kinematics of the particles we have that $q = p' - p \Rightarrow q^2 = -2(pq)$. From the Feynman paramteres formalism we know that $x+y+z=1$. Using these identities we arrive at the final equation for the denominator:

$$D = \left[l^2-(1-z)^2m^2+xyq^2+i\varepsilon\right]^3 = \left[l^2-\Delta+i\varepsilon\right]^3,\ \ \Delta = m^2(1-z)^2 - xyq^2 \tag{2.111}$$

We omit terms linear in l as they integrate to 0. Then

$$N^\mu = -2m^2\gamma^\mu + 4m(2k+p+p')^\mu - 2(\hat{k}+\hat{p})\gamma^\mu(\hat{k}+\hat{p}') =$$
$$-2m^2\gamma^\mu + 4m\left[2l+(1-2x)p+(1-2y)p'\right]^\mu - 2\hat{l}\gamma^\mu\hat{l} - 2[(1-x)\hat{p}-y\hat{p}']\gamma^\mu[(1-y)\hat{p}'-x\hat{p}]$$

We will consider each term separately and will simplify the whole expression as much as possible.

$$-2(y\hat{p}+z\hat{p}-y\hat{p}')\gamma^\mu(-x\hat{p}+x\hat{p}'+z\hat{p}') \to y = 1-x-z \to$$

$$(x-1)\hat{q}+z\hat{p}' \tag{2.112}$$

Next, we will write $-x\hat{p}+(1-y)\hat{p}'$ as

$$z\hat{p}+(1-y)\hat{q} \tag{2.113}$$

Therefore, the numerator equation becomes

$$\begin{aligned} N^\mu = -2m^2\gamma^\mu + 4m[z(p+p')+(x-y)q]^\mu - \\ -2\hat{l}\gamma^\mu\hat{l} - 2[(x-1)\hat{q}+z\hat{p}']\gamma^\mu[z\hat{p}+(1-y)\hat{q}] \end{aligned} \tag{2.114}$$

But the whole numerator is sandwiched between $\bar{u}(p')$ and $u(p)$. Therefore, we can apply Dirac's equation and few other identities:

$$\hat{p}u(p) = mu(p)$$
$$\{\hat{q},\gamma^\mu\} = 2q^\mu$$
$$\gamma^\mu\hat{q} = q^\mu + i\sigma^{\mu\nu}q_\nu$$
$$\bar{u}(p')[\hat{q}\gamma^\mu\hat{q}]u(p) = -q^2\bar{u}(p')\gamma^\mu u(p)$$

to get

$$\begin{aligned} \gamma^\mu[zm+(1-y)\hat{q}] = mz(x-1)\hat{q}\gamma^\mu + (1-x)(1-y)\hat{q}\gamma^\mu\hat{q} + m^2z^2\gamma^\mu + z(1-y)m\gamma^\mu\hat{q} = \\ mz[(x-1)q^\mu + z(1-y)m\gamma^\mu\hat{q} = mz[(x-1)\hat{q}\gamma^\mu + (1-y)\gamma^\mu\hat{q}] = \\ mz[(x-1)q^\mu - i(x-1)\sigma^{\mu\nu}q_\nu + (1-y)q^\mu + i(1-y)\sigma^{\mu\nu}q_\nu] = \\ mz[(x-y)q_\mu + i(2-x-y)\sigma^{\mu\nu}q_\nu] \end{aligned} \tag{2.115}$$

$$N^{\mu} = -2m^2\gamma^{\mu} + 4m[(1-2x)p + (1-2y)p']^{\mu} - 2\hat{l}\gamma^{\mu}\hat{l} +$$
$$+2mz(y-x)q^{\mu} - 2m^2z^2\gamma^{\mu} - 2(1-x)(1-y)q^2\gamma^{\mu} + 2mz(x+y-2)i\sigma^{\mu\nu}q_{\nu} \tag{2.116}$$

From the relation between x, y and z we have that $1-2x = y+z-x$ and $1-2y = x+z-y$. Therefore,

$$N^{\mu} = -2m^2\gamma^{\mu} + 4m[z(p+p') + (x-y)q]_{\mu} - 2\hat{l}\gamma^{\mu}\hat{l} -$$
$$-2\left[mz(x-y)q^{\mu} - imz(x+y-2)\sigma^{\mu\nu}q_{\nu} + m^2z^2\gamma^{\mu} + (1-x)(1-y)q^2\gamma^{\mu}\right] \tag{2.117}$$

Using Gordon's identity we get

$$-2m^2\gamma^{\mu} + 4m\left[8mz\left(\gamma^{\mu} - \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\right) + (x-y)q_{\mu}\right] - 2\hat{l}\gamma^{\mu}\hat{l} -$$
$$-2\left[mz(x-y)q^{\mu} - imz(x+y-2)\sigma^{\mu\nu}q_{\nu} + m^2z^2\gamma^{\mu} + (1-x)(1-y)q^2\gamma^{\mu}\right] \tag{2.118}$$

After grouping all terms we get that

$$\Gamma^{\mu}(p,p') = \mu^{4-d}\int \frac{d^dl}{(2\pi)^d}\int_0^1)\frac{2\delta(x+y+z-1}{[l^2-\Delta+i\varepsilon]^3}$$
$$\Bigg[-2\hat{l}\gamma^{\mu}\hat{l} + 2m(x-y)(2-z)q^{\mu} - 2(1-x)(1-y)q^2\gamma^{\mu} - 2m^2(z^2-4z+1)\gamma^{\mu}$$
$$-4m^2z(1-z)\frac{i\sigma^{\mu\nu}q_{\nu}}{2m} - 2m^2z^2\gamma^{\mu}\Bigg] \tag{2.119}$$

Due to Ward identity the term proportional to q^{μ} should vanish. Finally,

$$\Gamma^{\mu}(p,p') = 4ie^2\mu^{4-d}\int \frac{d^dl}{(2\pi)^d}\int_0^1 dxdydz\delta(x+y+z-1)\frac{1}{D}$$

$$\left\{\gamma^{\mu}\left(-\frac{1}{2}l^2 + (1-x)(1-y)q^2 + (1-4z+z^2)m^2\right) + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}\left(2m^2z(1-z)\right)\right\} \tag{2.120}$$

Again we have two parts of the integral - one with a numerator proportional to l^2 and one with a numerator independent of l. It should also be noted that the decomposition of the

Lagrangian into form factors is manifest - the term proportional to $\sigma^{\mu\nu}q_\nu$ is the first order correction to the g-factor, which is roughly 2. The first integral is

$$-2ie^2\int\frac{d^dl}{(2\pi)^d}\gamma^\mu\frac{l^2}{(l^2-\Delta+i\varepsilon)^3} \tag{2.121}$$

The Wick rotation calculation will give us

$$\frac{2e^2\gamma^\mu}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma\left(2-\frac{d}{2}\right)}{2}\frac{1}{(-\Delta)^{2-d/2}}\rightarrow\frac{e^2\gamma^\mu}{4\pi^2\varepsilon} \tag{2.122}$$

The other part of the integral will be

$$4ie^2\int\frac{d^dl}{(2\pi)^d}\frac{[...]}{(l^2-\Delta)^3}=4e^2[...]\int_0^\infty\frac{d^dl_E}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^3}\rightarrow\frac{ie^2}{8\pi^2}[...]\frac{1}{\Delta} \tag{2.123}$$

which in fact is a finite result. But performing the integration over the Feynman creates problems. In the limit $q\rightarrow 0$ the integration gives

$$\int_0^1 dxdydz\delta(x+y+z-1)\frac{e^2\gamma^\mu}{4\pi^2\varepsilon}=\frac{e^2\gamma^\mu}{8\pi^2\varepsilon} \tag{2.124}$$

$$\frac{ie^2\gamma^\mu}{8\pi^2}\int dxdydz\frac{1-4z+z^2}{(1-z)^2}\delta(x+y+z-1)\rightarrow\text{Divergent} \tag{2.125}$$

$$\frac{e^2}{8\pi^2}\frac{i\sigma^{\mu\nu}q_\nu}{2m}\int\frac{2z}{1-z}\delta(x+y+z-1)=\frac{\alpha}{2\pi} \tag{2.126}$$

which is the correction to the Dirac g-factor and was first calculated by Julian Schwinger in 1948.

In order to subtract all those infinities, which arise in the loop diagrams, we introduce counter terms in the Lagrangian, which eactly cancel these infinities. The original Lagrangian of quantum electrodynamics is

$$\mathscr{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(i\hat{\partial}-m_0)\psi-e_0\bar{\psi}\gamma^\mu\psi A_\mu \tag{2.127}$$

Let also define the self energy diagram to get a correction $\frac{iZ_2}{\hat{p}-m}$ and the vacuum polarization diagram - $\frac{iZ_3g_{\mu\nu}}{q^2}$. Lets also define $\psi=Z_2^{1/2}\psi_r$ and $A^\mu=z_3^{1/2}A_r^\mu$, where the index r stands for regularized. Then the regularized Lagrangian becomes

$$\mathscr{L} = -\frac{1}{4}Z_3(F_r^{\mu\nu})^2 + Z_2\bar{\psi}_r(i\hat{\partial} - m_0) - e_0 Z_2 Z_3^{1/2}\bar{\psi}_r\gamma^\mu\psi_r A_{r\mu} \tag{2.128}$$

The quantities with an index 0 are "bare" and unobservable, whereas the observable mass and charge are denoted by m and e. We can split the Lagrangian into two parts - bare and observable

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}(F_r^{\mu\nu})^2 + \bar{\psi}_r(i\hat{\partial} - m)\psi_r - e\bar{\psi}_r\gamma^\mu\psi A_\mu - \\ &-\frac{1}{4}\delta_3(F_r^{\mu\nu})^2 + \bar{\psi}_r(i\hat{\partial} - \delta m)\psi_r - e\delta_1\bar{\psi}_r\gamma^\mu\psi A_\mu,\end{aligned} \tag{2.129}$$

where $\delta_m = Z_2 m_0 - m$ and $\delta_1 = Z_2 - 1$. In order for the above Lagrangian to lead to convergent interactions, the matrix elements for the divergent processes considered above need to obey the following conditions:

$$\begin{aligned} M(\hat{p} = m) &= 0 \\ \left.\frac{dM(\hat{p})}{d\hat{p}}\right|_{\hat{p}=m} &= 0 \\ \Pi(q^2 = 0) &= 0 \\ -ie\Gamma^\mu(p,p') &= -ie\gamma^\mu \end{aligned} \tag{2.130}$$

Chapter 3

SU(2) Nambu-Jona-Lasinio Model with a Massless Quark

In this chapter we consider the foundations of our later work. The theory developed is based on the work of A.Osipov [27] and M. Chizhov [26], where the first paper is a classic work on $SU(2)$ NJL model with a massless quark, whereas the second paper is expanding it by introducing tensor interactions and therefore predicting a novel mass relation between the meson excitations within this model. The effective Lagrangians which are discussed in both papers include only local Lorenz invariant terms without derivatives. This allows us to expand vector interactions into tensor interactions by sandwiching the tensor $\sigma_{\mu\nu}$ with two vector fields, which is done in the second paper. We will show the derivations of the main results and will highlight how the two papers complement each other. All results are published in [36] and [37].

3.1 SU(2) Lagrangian without Tensor Interactions

We start the analysis of the model by first deriving the effective Lagrangian from the work of Osipov and then completing it by adding the terms for tensor interactions. In order to account for the meson excitations we bozonize the four-fermion interaction written as equation 3.1 by using the bozonization functional relation 3.2:

$$\mathscr{L} = \bar{\Psi}\hat{q}\Psi + \frac{G_0}{2}\left[(\bar{\Psi}\Psi)^2 - (\bar{\Psi}\gamma^5\vec{\tau}\Psi)^2\right] - \frac{G_V}{2}\left[\bar{\Psi}\gamma_\mu\vec{\tau}\Psi\right]^2 - \frac{G_A}{2}\left[\bar{\Psi}^{\gamma}_{\mu}\gamma^5\vec{\tau}\Psi\right]^2 \tag{3.1}$$

$$\exp\left[-\frac{i}{2}J\mathscr{K}^{-1}J\right] = \int [d\phi]\exp\left[iJ\phi + \frac{i}{2}\phi\mathscr{K}\phi\right] \tag{3.2}$$

After bozonization we define the meson fields as

$$\begin{aligned} \sigma = \frac{G_0}{g_\sigma}\bar{\Psi}\Psi,\ \ \vec{\pi} = i\frac{G_0}{g_\pi}\bar{\Psi}\gamma^5\vec{\tau}\Psi \\ \vec{V}_\mu = -\frac{G_V}{g_V}\bar{\Psi}\gamma_\mu\vec{\tau}\Psi,\ \ \vec{A}_\mu = -\frac{G_A}{g_A}\bar{\Psi}\gamma_\mu\gamma^5\vec{\tau}\Psi \end{aligned} \tag{3.3}$$

and the linearized Lagrangian is

$$\begin{aligned} \mathscr{L} = \bar{\Psi}\hat{q}\Psi + g_\sigma\bar{\Psi}\Psi\sigma - \frac{g_\sigma^2}{2G_0}\sigma^2 + ig_\pi\bar{\Psi}\gamma^5\vec{\tau}\Psi\vec{\pi} - \frac{g_\pi^2}{2G_0}\pi^2 + \\ + g_V\bar{\Psi}\gamma_\mu\vec{\tau}\Psi\vec{V}_\mu + \frac{g_V^2}{2G_V}\vec{V}_\mu^2 + g_A\bar{\Psi}\gamma_\mu\gamma^5\vec{\tau}\Psi\vec{A}_\mu + \frac{g_A^2}{2G_A}\vec{A}_\mu^2 \end{aligned} \tag{3.4}$$

One can note that in the Lagrangian 3.4 we have scalar and vector sector with pseudo-scalar/vector interactions. From here we can deduce the Feynman rules for the theory

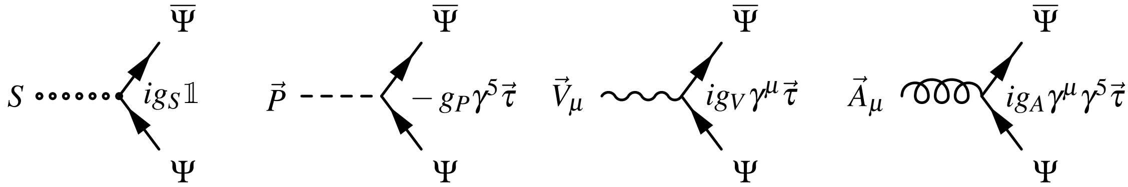


Fig. 3.1 Basic Feynman diagrams for the linearized meson fields.

The definitions of the meson fields can be used to calculate the quantum numbers of the particles, all listed in the table below

collective meson states:	S	$\vec{P}$	$\vec{V}_\mu$	$\vec{A}_\mu$
quantum numbers $I^G(J^{PC})$:	$0^+(0^{++})$	$1^-(0^{-+})$	$1^+(1^{--})$	$1^-(1^{++})$

Table 3.1 The quantum numbers of the collective meson states.

The mathematical form of the interactions included are purely classical. In order to quantize them we calculate all Feynman diagrams and therefore we can write an effective quantum theory; we will restrict ourselves to one-loop order. The first class of diagrams to be calculated are self-energy diagrams for each of the mesons.

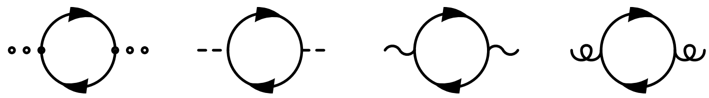


Fig. 3.2 These are a self-energy diagrams for the σ, π, V_μ and A_μ.

Complete derivation for the σ meson self energy will be given, the rest of the diagrams are obtained by analogy and only final results would be listed. The amplitude is obtained by the Feynman rules

$$\Pi(q) = ig_\sigma^2 N_c \mathrm{Tr}[\mathbb{1}] \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[(\hat{p} - m_0)^{-1}(\hat{p} - \hat{q} - m_0)^{-1}\right], \tag{3.5}$$

where N_c is the number of the colour charges and m_0 is the quark mass which is taken to be 0. Expanding the trace brackets the non zero terms are

$$\Pi(q) = 8ig_\sigma^2 N_c \int \frac{d^4p}{(2\pi)^4} \frac{p^2 - (pq)}{(p^2 - m_0^2)\left[(p-q)^2 - m_0^2\right]}. \tag{3.6}$$

We note that the numerator is of order p^6 and the denominator is of order p^4, so the whole integral is quadratically divergent. Expanding the denominator as an arithmetic series

$$\frac{1}{(p^2 - m_0^2)\left[(p-q)^2 - m_0^2\right]} = \frac{1}{(p^2 - m_0^2)^2}\left\{1 + \frac{q(2p-q)}{p^2 - m_0^2} + \left[\frac{q(2p-q)}{p^2 - m_0^2}\right]^2\right\}, \tag{3.7}$$

where we retain only terms which are divergent. It should be also noted that integrals over terms which are of odd power in p give no contribution. All diagrams which will be computed in this thesis involve two divergent integrals and this notation will be common for the rest of the chapters.

$$I_0 = -i \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - m_0^2)^2} \text{ and } I_2 = i \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m_0^2} \tag{3.8}$$

The final result written in terms of these integrals is

$$\Pi_\sigma(q) = 8g_\sigma N_c I_2 + 4g_\sigma^2 N_c I_0 q^2. \tag{3.9}$$

Since all results need to be normalized and finite we have

$$4g_\sigma^2 N_c I_0 = 1 \tag{3.10}$$

We repeat the same procedure for the π field

$$\begin{aligned}\Pi_\pi(q) = -g_\pi^2 i^2(-i)N_c\mathrm{Tr}[\tau_i\tau_j]\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^5(\hat{p}+m_0)\gamma^5(\hat{p}-\hat{q}+m_0)\right]}{(p^2-m_0^2)\left[(p-q)^2-m_0^2\right]} = \\ = 8g_\pi^2 N_c I_2\delta_{ij} + 4g_\pi^2 N_c I_0 q^2\delta_{ij}\end{aligned} \tag{3.11}$$

For the vector field we have the following result

$$\begin{aligned}\Pi_V^{\mu\nu}(q) = ig_V^2 N_c\mathrm{Tr}[\tau_i\tau_j]\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}[\gamma^\mu(\hat{p}+m_0)\gamma^\nu(\hat{p}-\hat{q}+m_0)]}{(p^2-m_0^2)\left[(p-q)^2-m_0^2\right]} = \\ = -4g_V^2 N_c g_{\mu\nu} I_2\delta_{ij} + \frac{8}{3}g_V^2 N_c(q_\mu q_\nu - q^2 g_{\mu\nu})I_0\delta_{ij}\end{aligned} \tag{3.12}$$

For the axial-vector field we have

$$\begin{aligned}\Pi_A^{\mu\nu}(q) = ig_A^2 N_c\mathrm{Tr}[\tau_i\tau_j]\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\mu\gamma^5(\hat{p}+m_0)\gamma^\nu\gamma^5(\hat{p}-\hat{q}+m_0)\right]}{(p^2-m_0^2)\left[(p-q)^2-m_0^2\right]} = \\ = -4g_A^2 N_c g_{\mu\nu} I_2\delta_{ij} + \frac{8}{3}g_A^2 N_c(q_\mu q_\nu - q^2 g_{\mu\nu})I_0\delta_{ij}\end{aligned} \tag{3.13}$$

In $SU(2)$ loop diagrams which include external lines corresponding to different particles are 0, since $\mathrm{Tr}[\tau_i\tau_j] = 2\delta_{ij}$; in $U(1)$ are also 0 either due to trace of an odd number of Dirac matrices, or due to symmetry. This guarantees that at this level there is no mixing between any particles. Applying condition 3.10 to all results we obtain a connection between all interaction constants and therefore, the whole theory can be described by a single constant:

$$3g_\sigma^2 = 3g_\pi^2 = 2g_V^2 = 2g_A^2 = \frac{3}{4N_c I_0} \tag{3.14}$$

3.2 Diagrams with Three External Lines

In this section we will calculate the non-zero diagrams including three external lines, or triangular loop diagrams. The momentum nomenclature that will be conventional throughout the whole section is

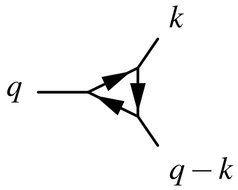


Fig. 3.3 Momentum notation convention in triangular diagrams that will be used for the rest of the chapter.

We have 4 particles and three external lines, which makes 20 diagrams in total. Among all of these diagrams we are left with 4 due to the properties of τ matrices or symmetry arguments. The first non-zero diagram we will consider is with σ, π and A_μ.

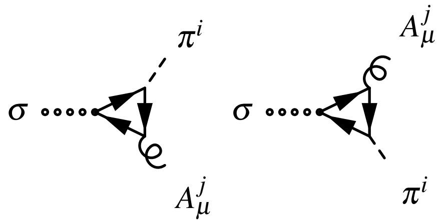


Fig. 3.4 Interaction diagram between σ, π and A_μ.

$$
\begin{aligned}
&g_\sigma g_\pi g_A \mathrm{Tr}[\tau_{ij}]N_c \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\left[(\hat{p}-\hat{q})\gamma_\mu\gamma^5(\hat{p}-\hat{k})\gamma^5\hat{p}\right]}{(p^2-m_0^2)\left[(p-q)^2-m_0^2\right]\left[(p-k)^2-m_0^2\right]} + \\
&+g_\sigma g_\pi g_A \mathrm{Tr}[\tau_{ij}]N_c \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\hat{p}\gamma^5(\hat{p}+\hat{k})\gamma_\mu(\hat{p}+\hat{q})\right]}{(p^2-m_0^2)\left[(p+q)^2-m_0^2\right]\left[(p+k)^2-m_0^2\right]}
\end{aligned}
\tag{3.15}
$$

In this case the denominator is of order p^6 and the numerator is of maximal order p^7 meaning that we need to expand the denominator until we stop getting divergent terms. The final result is

$$
\mathscr{L}_{\sigma\pi A} = 8ig_\sigma g_\pi g_A N_c \delta_{ij}(k+q)^\mu I_0 \tag{3.16}
$$

The next non-zero diagram is $V_\mu\pi\pi$

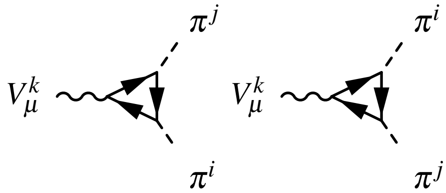


Fig. 3.5 Interaction diagram between a vector meson and two pions.

$$-\varepsilon^{ijk} g_V g_\pi^2 N_c \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu (\hat{p}-\hat{q})(\hat{p}-\hat{k})\hat{p}\right]}{(p^2-m_0^2)\left[(p-q)^2-m_0^2\right]\left[(p-k)^2-m_0^2\right]} + \\ +\varepsilon^{ijk} g_V g_\pi^2 N_c \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu \hat{p}(\hat{p}+\hat{k})(\hat{p}+\hat{q})\right]}{(p^2-m_0^2)\left[(p+q)^2-m_0^2\right]\left[(p+k)^2-m_0^2\right]} \tag{3.17}$$

or after simplification the effective Lagrangian becomes

$$\mathscr{L}_{V\pi\pi} = 8i\varepsilon^{ijk} g_V g_\pi^2 N_c \left(\frac{1}{2}q^\mu - k^\mu\right) I_0 \tag{3.18}$$

The next diagram is with three vector mesons

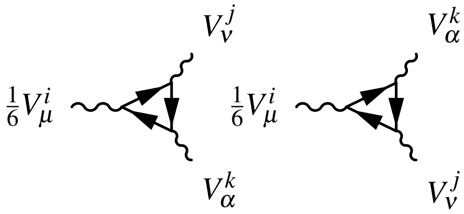


Fig. 3.6 Interaction diagram between three vector mesons.

Applying Feynman rules to these diagrams we get

$$-\frac{1}{6}2\varepsilon^{ijk} g_V^3 N_c \int \frac{d^4p}{(2pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu (\hat{p}-\hat{q})\gamma^\alpha(\hat{p}-\hat{k})\gamma^\nu \hat{p}\right]}{(p^2-m_0^2)\left[(p-q)^2-m_0^2\right]\left[(p-k)^2-m_0^2\right]} + \\ +\frac{1}{6}2\varepsilon^{ijk} g_V^3 N_c \int \frac{d^4p}{(2pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu \hat{p}\gamma^\nu(\hat{p}+\hat{k})\gamma^\alpha(\hat{p}+\hat{q})\right]}{(p^2-m_0^2)\left[(p+q)^2-m_0^2\right]\left[(p+k)^2-m_0^2\right]} \tag{3.19}$$

or finally

$$\mathscr{L}_{VVV} = \frac{8}{9} i g_V^3 N_c \varepsilon^{ijk} \left[-g^{\mu\nu}(k+q)^\alpha + (2q-k)^\nu g^{\alpha\mu} + (2k-q) g^{\alpha\nu}\right] I_0 \tag{3.20}$$

The final non-zero diagram with three external lines is one including a vector meson and two axial vector mesons.

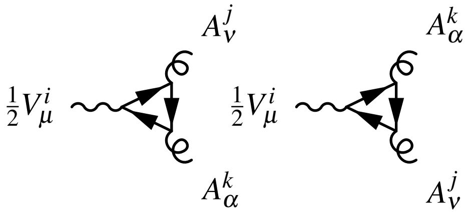


Fig. 3.7 Interaction diagram between a vector meson and two axial mesons.

$$-\frac{1}{2}g_A^2 g_V 2\varepsilon^{ijk}N_c\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\mu(\hat{p}-\hat{q})\gamma_\alpha\gamma^5(\hat{p}-\hat{k})\gamma_\nu\gamma^5\hat{p}\right]}{(p^2-m_0^2)\left[(p-q)^2-m_0^2\right]\left[(p-k)^2-m_0^2\right]}+$$
$$+-\frac{1}{2}g_A^2 g_V 2\varepsilon^{ijk}N_c\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}\left[\gamma^\mu\hat{p}\gamma^\nu\gamma^5(\hat{p}+\hat{k})\gamma^\alpha\gamma^5(\hat{p}+\hat{q})\right]}{(p^2-m_0^2)\left[(p+q)^2-m_0^2\right]\left[(p+k)^2-m_0^2\right]} \quad (3.21)$$

For the sake of compactness we introduce the variables $q_1=-q$, $q_2=k$ and $q_3=q-k$. Then written in terms of these variables the final answer is

$$\mathscr{L}_{VAA}=\frac{8}{3}iN_c g_A^2 g_V\varepsilon^{ijk}\left[(q_3-q_1)^\nu g^{\alpha\mu}+(q_1-q_2)^\alpha g^{\mu\nu}+(q_2-q_3)^\mu g^{\alpha\nu}\right]I_0 \quad (3.22)$$

3.3 Box Feynman Diagrams

In this section we consider box diagrams, or diagrams with four external lines. Because of Dirac matrix algebra, or on symmetry grounds most of the diagrams equate to 0 and here we will lay out the derivations of the non-zero effective Lagrangians. The first diagram is with four σ mesons

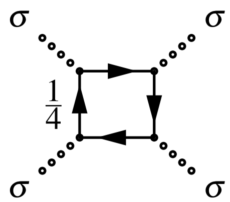


Fig. 3.8 Box diagram with four σ mesons.

The corresponding effective Lagrangian is

$$\mathscr{L}_{SSSS}=\frac{1}{4}ig_\sigma^4 N_c\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}\left[\hat{p}\hat{p}\hat{p}\hat{p}\right]}{p^8}=-g_\sigma^4 N_c\mathrm{Tr}[\mathbb{1}]I_0 \quad (3.23)$$

The next diagram is with two σ mesons and two π mesons:

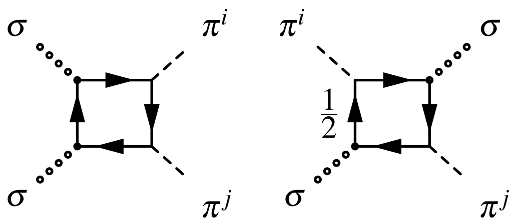


Fig. 3.9 Box diagram with two σ mesons and two π mesons.

Applying the Feynman rules we get

$$\mathscr{L}_{\sigma\sigma\pi\pi} = -4g_\sigma^2 g_\pi^2 N_c \delta^{ij} I_0 \tag{3.24}$$

The next diagram contains two σ mesons and two axial-vector mesons:

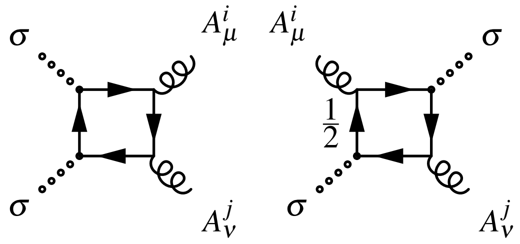


Fig. 3.10 Box diagram with two σ mesons and two A_μ mesons.

The effective Lagrangian of this diagram is

$$\mathscr{L}_{\sigma\sigma AA} = 8g_\sigma^2 g_A^2 N_c \delta^{ij} g^{\mu\nu} I_0 \tag{3.25}$$

The next diagram we consider contains all four mesons

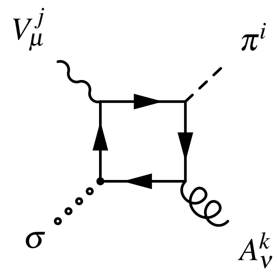


Fig. 3.11 Box diagram containing all four mesons at once.

The effective Lagrangian reads

$$\mathscr{L}_{\sigma\pi AV} = -ig_\sigma g_\pi g_V g_A N_c I_0 \mathrm{Tr}\left(\left[V_\mu, \pi\right]\{A_\nu, \sigma\} + \{A_\nu, \sigma\}\left[V_\mu, \pi\right]\right) \tag{3.26}$$

The next diagram contains only π mesons

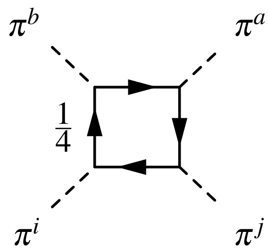


Fig. 3.12 Box diagram containing only pions.

The resultant effective Lagrangian is

$$\mathscr{L}_{\pi\pi\pi\pi} = -g_\pi^4 N_c \mathrm{Tr}(\tau^i \tau^j \tau^a \tau^b) I_0 \tag{3.27}$$

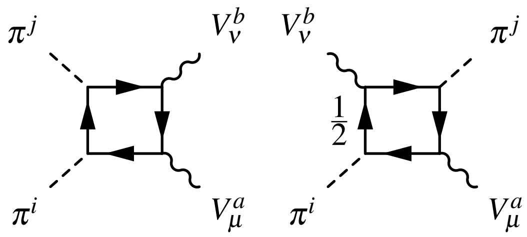


Fig. 3.13 Box diagram with two σ mesons and two V_μ mesons.

and the corresponding Lagrangian

$$\mathscr{L}_{\pi\pi VV} = 8 g_\pi^2 g_V^2 \left[\vec{V}_\mu^2 \vec{\pi}^2 - (\vec{V}_\mu \vec{\pi})^2 \right] \tag{3.28}$$

The next diagram contains two π mesons and two axial mesons

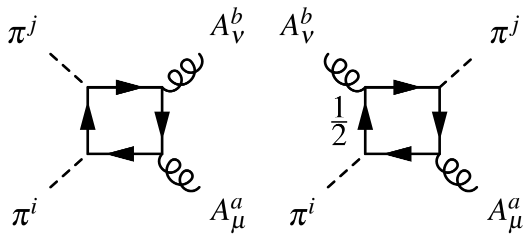


Fig. 3.14 Box diagram with two π mesons and two A_μ mesons.

$$\mathscr{L}_{\pi\pi AA} = \frac{1}{2} g_\pi^2 g_A^2 N_c I_0 \mathrm{Tr}\left[\{A_\mu, \pi\}\{A_\nu, \pi\} \right] \tag{3.29}$$

The next diagram we consider contains four vector mesons

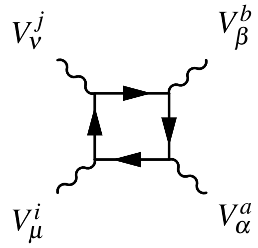


Fig. 3.15 Box diagram with four V_μ

$$\mathscr{L}_{VVVV} = -\frac{1}{3} g_V^4 N_c \mathrm{Tr}\left([V_\mu, V_\nu][V_\mu, V_\nu] \right) \tag{3.30}$$

For the diagram with four axial-vector mesons we have the same structure of the dynamics, except for the coefficient

$$\mathscr{L}_{AAAA} = -\frac{1}{3} g_A^4 N_c \mathrm{Tr}\left([A_\mu, A_\nu][A_\mu, A_\nu]\right) \tag{3.31}$$

The final diagram contains two vector and two axial-vector mesons:

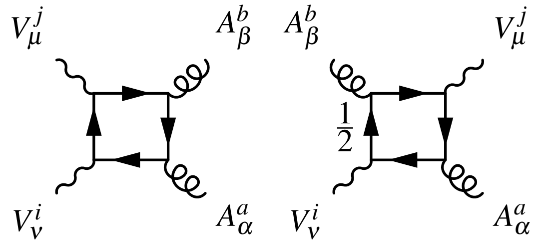


Fig. 3.16 Box diagram with two V_μ mesons and two A_μ mesons.

and the effective Lagrangian is

$$\begin{aligned}\mathscr{L}_{VVAA} = &-\frac{4}{3} g_V^2 g_A^2 N_c \mathrm{Tr}(\tau^i \tau^j \tau^a \tau^b)\left(-2g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\right) I_0 - \\ &-\frac{2}{3} g_V^2 g_A^2 N_c \mathrm{Tr}(\tau^i \tau^a \tau^j \tau^b)\left(-2g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu}\right) I_0\end{aligned} \tag{3.32}$$

Collecting all terms we obtain a complete effective Lagrangian describing an $SU(2)$ NJL model including (pseudo-)scalar and (pseudo-)vector fields. The resulting Lagrangian is identical to the one presented in the work of A. Osipov [27] except for the terms arising from the vector and axial-vector self interactions. The missing terms are

$$\left(\frac{1}{4G_V} - g_V^2 N_c I_2\right) \mathrm{Tr}\left[V_\mu^2\right] + \left(\frac{1}{4G_A} - g_A^2 N_c I_2\right) \mathrm{Tr}\left[A_\mu^2\right] \tag{3.33}$$

3.4 SU(2) NJL Model with Tensor Fields

In this section we consider an $SU(2)$ model including meson interactions described by tensor currents. As it was stated earlier we will not include terms involving derivatives of fields which leaves us with only two local Lorentz invariant tensor interactions. The Lagrangian 3.1 becomes

$$\begin{aligned}\mathscr{L} = \bar{\Psi}\hat{q}\Psi + \frac{G_0}{2}\left[(\bar{\Psi}\Psi)^2 - (\bar{\Psi}\gamma^5\vec{\tau}\Psi)^2\right] - \frac{G_V}{2}\left[\bar{\Psi}\gamma_\mu\vec{\tau}\Psi\right]^2 - \frac{G_A}{2}\left[\bar{\Psi}^{\gamma}_{\mu}\gamma^5\vec{\tau}\Psi\right]^2 - \\ - \frac{G_T}{2}\bar{\Psi}\sigma_{\mu\lambda}\left(1+\gamma^5\right)\vec{\tau}\Psi\frac{q^\mu q_\nu}{q^2}\left(1-\gamma^5\right)\vec{\tau}\Psi\end{aligned} \tag{3.34}$$

After bozonization we have four additional terms to 3.4 describing the dynamics of the tensor fields

$$\begin{aligned}\mathscr{L} = \bar{\Psi}\hat{q}\Psi + g_\sigma\bar{\Psi}\Psi\sigma - \frac{g_\sigma^2}{2G_0}\sigma^2 + ig_\pi\bar{\Psi}\gamma^5\vec{\tau}\Psi\vec{\pi} - \frac{g_\pi^2}{2G_0}\pi^2 + \\ + g_V\bar{\Psi}\gamma_\mu\vec{\tau}\Psi\vec{V}_\mu + \frac{g_V^2}{2G_V}\vec{V}_\mu^2 + g_A\bar{\Psi}\gamma_\mu\gamma^5\vec{\tau}\Psi\vec{A}_\mu + \frac{g_A^2}{2G_A}\vec{A}_\mu^2 - \\ - ig_R\bar{\Psi}\sigma_{\mu\nu}\vec{\tau}\Psi\frac{q_\mu}{|q|}\vec{R}_\nu + \frac{g_R^2}{2G_T}\vec{R}_\mu^2 + g_B\bar{\Psi}\sigma_{\mu\nu}\gamma^5\Psi\frac{q_\mu}{|q|}\vec{B}_\nu + \frac{g_B^2}{2G_T}\vec{B}_\mu^2\end{aligned} \tag{3.35}$$

From here we can establish the Feynman rules regarding the tensor fields

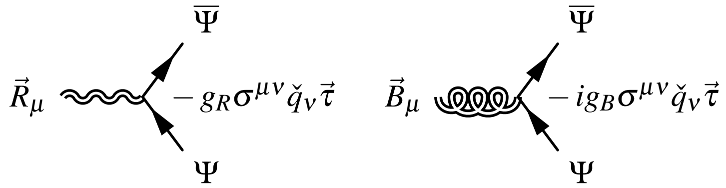


Fig. 3.17 Basic Feynman diagrams for the tensor collective modes after linearization. We use the notation $\check{q}_\nu = \frac{q_\nu}{|q|}$.

Once we have the Feynman rules we repeat the procedure from the previous sections in order to obtain quantum corrections to the auxiliary fields which would result in distinguishing the kinetic and mass terms. We first calculate the loop diagrams (self-energies)

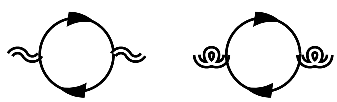


Fig. 3.18 These are a self-energy diagrams for the $\vec{R}_\mu$ and $\vec{B}_\mu$.

For the R field the self energy diagram gives

$$\begin{aligned}\Pi^R_{\mu\nu} = -i\frac{g_R^2}{q^2}N_c\mathrm{Tr}(\tau^i\tau^j)\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{Tr}\left[\sigma_{\mu\alpha}q_\alpha(\hat{p}-\hat{q})\sigma_{\nu\beta}q_\beta\hat{p}\right]}{(p^2-m^2)\left[(p-q)^2-m^2\right]} = \\ = -\frac{4}{3}g_R^2N_c\mathrm{Tr}(\tau^i\tau^j)\left(q^\mu q^\nu - q^2g^{\mu\nu}\right)I_0\end{aligned} \tag{3.36}$$

For the B field we get a similar result

$$\Pi^B_{\mu\nu} = -\frac{4}{3} g_B^2 N_c \mathrm{Tr}(\tau^i \tau^j) \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) I_0 \tag{3.37}$$

Due to the property $\mathrm{Tr}(\tau_i \tau_j) = 2\delta_{ij}$ there is no mixing between the R and B field. Therefore

$$\mu_T = \mu_R = \mu_B = \frac{g_T^2}{G_T} \tag{3.38}$$

Next we consider triangular diagrams involving tensor fields. The first non-zero diagram includes V, R and σ meson

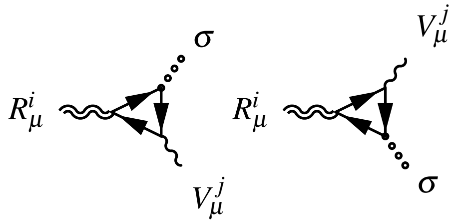


Fig. 3.19 Interaction diagram between σ, $\vec{R}_\mu$ and $\vec{V}_\mu$.

$$\begin{aligned}
\mathscr{L}_{RVS} = & \mathrm{Tr}(\tau^i \tau^j) g_R \frac{q_\nu}{|q|} g_\sigma g_V N_c \int \frac{d^4 p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma^{\mu\nu}(\hat{p}-\hat{q})\gamma_\alpha(\hat{p}-\hat{k})\hat{p}\right]}{[(p-q)^2-m^2]\,[(p-k)^2-m^2]\,(p^2-m^2)} + \\
& + \mathrm{Tr}(\tau^i \tau^j) g_R \frac{q_\nu}{|q|} g_\sigma g_V N_c \int \frac{d^4 p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma^{\mu\nu}\hat{p}(\hat{p}+\hat{k})\gamma_\alpha(\hat{p}+\hat{q})\right]}{[(p+q)^2-m^2]\,[(p+k)^2-m^2]\,(p^2-m^2)} = \\
& = 4 g_V g_\sigma g_R \mathrm{Tr}(\tau^i \tau^j) N_c \frac{1}{|q|} I_0 \left[-q^\mu q^\alpha + q^2 g^{\mu\nu} + k^\mu q^\alpha - (kq) g^{\alpha\nu}\right]
\end{aligned} \tag{3.39}$$

Next we have and interaction diagram between R, π and A meson.

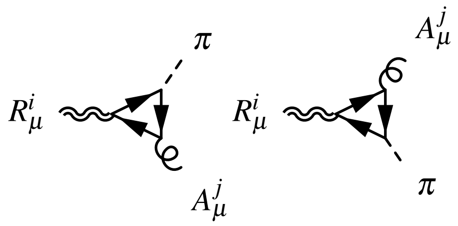


Fig. 3.20 Interaction diagram between σ, $\vec{R}_\mu$ and $\vec{V}_\mu$.

$$\mathscr{L}_{RAP} = -\mathrm{Tr}(\tau^i\tau^k\tau^j) g_R \frac{q_\nu}{|q|} g_\pi g_A N_c \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma^{\mu\nu}(\hat{p}-\hat{q})\gamma_\alpha(\hat{p}-\hat{k})\hat{p}\right]}{[(p-q)^2-m^2]\,[(p-k)^2-m^2]\,(p^2-m^2)} +$$
$$+\mathrm{Tr}(\tau^i\tau^j\tau^k) g_R \frac{q_\nu}{|q|} g_\pi g_A N_c \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma^{\mu\nu}\hat{p}(\hat{p}+\hat{k})\gamma_\alpha(\hat{p}+\hat{q})\right]}{[(p+q)^2-m^2]\,[(p+k)^2-m^2]\,(p^2-m^2)} =$$
$$= 4iN_c g_R g_\pi g_A \frac{q^\nu}{|q|} \mathrm{Tr}(\tau^i\tau^j\tau^k) I_o \left[-q^\mu q^\alpha + q^2 g^{\alpha\mu} + k^\mu q^\alpha - (kq) g^{\alpha\mu}\right] \tag{3.40}$$

Then we have the mixing diagram between B and V meson

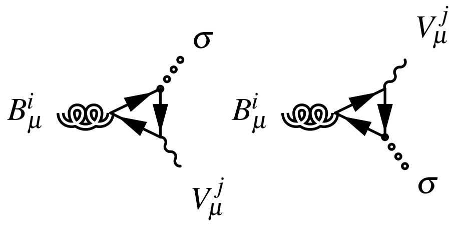


Fig. 3.21 Interaction diagram between σ, $\vec{B}_\mu$ and $\vec{V}_\mu$.

$$\mathscr{L} = \frac{1}{2} N_c g_\sigma g_V \frac{g_B}{|q|} \mathrm{Tr}(\tau^i\tau^j) \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu\nu}\gamma^5 q_\nu \hat{p}\gamma_\alpha(\hat{p}+\hat{k})(\hat{p}+\hat{q})\right]}{(p^2-m^2)\,[(p+k)^2-m^2]\,[(p+q)^2-m^2]} -$$
$$-\frac{1}{2} N_c g_\sigma g_V \frac{g_B}{|q|} \mathrm{Tr}(\tau^i\tau^j) \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu\nu}\gamma^5 q_\nu (\hat{p}-\hat{q})(\hat{p}-\hat{k})\gamma_\alpha \hat{p}\right]}{(p^2-m^2)\,[(p-k)^2-m^2]\,[(p-q)^2-m^2]} = \tag{3.41}$$
$$= 0$$

From the previous result we can conclude that we do not observe mixing between the B and the vector meson.

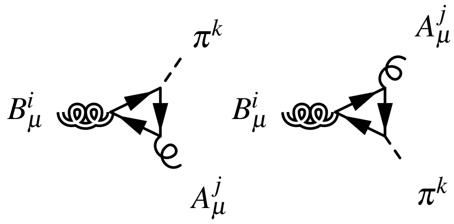


Fig. 3.22 Interaction diagram between π, $\vec{B}_\mu$ and $\vec{A}_\mu$.

$$\begin{aligned}\mathscr{L}_{BPA} = \frac{g_B}{|q|} g_\pi g_A N_c \mathrm{Tr}(\tau^i \tau^j \tau^k) \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu\nu}\gamma^5 q_\nu \hat{p}\gamma_\alpha(\hat{p}+\hat{k})(\hat{p}+\hat{q})\right]}{(p^2-m^2)\left[(p+k)^2-m^2\right]\left[(p+q)^2-m^2\right]} - \\ -\frac{g_B}{|q|} g_\pi g_A N_c \mathrm{Tr}(\tau^i \tau^k \tau^j) \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu\nu}\gamma^5 q_\nu(\hat{p}-\hat{q})(\hat{p}-\hat{k})\gamma_\alpha\hat{p}\right]}{(p^2-m^2)\left[(p+k)^2-m^2\right]\left[(p+q)^2-m^2\right]} = \\ = 4iN_c\frac{g_B}{|q|} g_\pi g_A \mathrm{Tr}[\tau^i,[\tau^j,\tau^k]] I_0 \varepsilon^{\alpha\mu kq}\end{aligned} \tag{3.42}$$

Now we proceed with diagrams involving two tensor particles.

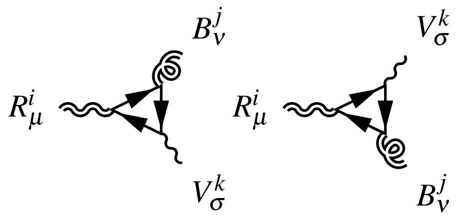


Fig. 3.23 Interaction diagram between $\vec{R}$, $\vec{B}$ and $\vec{V}$.

For the calculation of this diagram we will use the properties

$$\begin{aligned}\gamma^\alpha\gamma^\sigma\gamma_\alpha = -2\gamma^\sigma \\ \gamma^\alpha\sigma_{\mu\nu}\gamma^\alpha = 0\end{aligned} \tag{3.43}$$

$$\begin{aligned}\mathscr{L}_{RBV} = -\frac{g_R}{|q|}\frac{g_B}{|k|} g_V N_C \mathrm{Tr}[\tau^i\tau^k\tau^j] \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu q}(\hat{p}-\hat{q})\gamma_\sigma(\hat{p}-\hat{k})\sigma_{\nu k}\gamma^5\hat{p}\right]}{(p^2-m^2)\left[(p-q)^2-m^2\right]\left[(p-k)^2-m^2\right]} - \\ -\frac{g_R}{|q|}\frac{g_B}{|k|} g_V N_C \mathrm{Tr}[\tau^i\tau^j\tau^k] \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu q}\hat{p}\sigma_{\nu k}\gamma^5(\hat{p}+\hat{k})\gamma_\sigma(\hat{p}+\hat{q})\right]}{(p^2-m^2)\left[(p+q)^2-m^2\right]\left[(p+k)^2-m^2\right]} = \\ = \frac{2}{3}\frac{g_B}{|q|}\frac{g_R}{|k|} g_V N_c i I_0\Big[(k+q)^\mu\varepsilon^{\nu\sigma kq} + (k+q)^\nu\varepsilon^{\mu\sigma kq} + \\ +(k-q)^\sigma\varepsilon^{\mu\nu kq} + ((kq)+k^2)\varepsilon^{\mu\nu\sigma q} + ((kq)+q^2)\varepsilon^{\mu\nu\sigma k}\Big]\end{aligned} \tag{3.44}$$

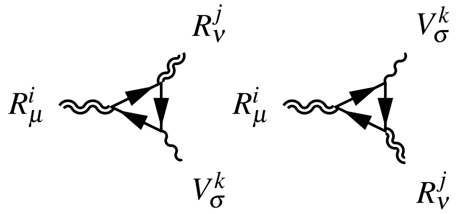


Fig. 3.24 Interaction diagram between two $\vec{R}$ and $\vec{V}$.

$$
\begin{aligned}
\mathscr{L}_{RRV} = \frac{1}{2} i \frac{g_R^2}{|q||k|} g_V N_c \mathrm{Tr}(\tau^i \tau^k \tau^j) \int \frac{d^4 p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu q}(\hat{p}-\hat{q})\gamma_\sigma(\hat{p}-\hat{k})\sigma_{\nu k}\hat{p}\right]}{(p^2-m^2)\left[(p-q)^2-m^2\right]\left[(p-k)^2-m^2\right]} + \\
-\frac{1}{2} i \frac{g_R^2}{|q||k|} g_V N_c \mathrm{Tr}(\tau^i \tau^j \tau^k) \int \frac{d^4 p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu q}\hat{p}\sigma_{\nu k}(\hat{p}+\hat{k})\gamma_\sigma(\hat{p}+\hat{q})\right]}{(p^2-m^2)\left[(p+q)^2-m^2\right]\left[(p+k)^2-m^2\right]} = \\
-\frac{1}{3} N_c \frac{g_R^2}{|q||k|} g_V \mathrm{Tr}[\tau^i[\tau^j,\tau^k]] I_o \Big[-q^\mu q^\nu k^\sigma - q^\alpha k^\mu k^\nu + \\
+ g^{\mu\nu}(k^\sigma k^2 + k^\sigma q^2) - g^{\mu\sigma}(q^\nu k^2 - k^\nu (kq)) + g^{\nu\sigma}(q^\mu (qk) - k^\mu q^2) \Big]
\end{aligned}
\tag{3.45}
$$

The result for the interaction diagram between two B mesons and a vector meson is similar except for the constants

$$
\begin{aligned}
\mathscr{L}_{BBV} = -\frac{1}{3} N_c \frac{g_B^2}{|q||k|} g_V \mathrm{Tr}[\tau^i[\tau^j,\tau^k]] I_o \Big[-q^\mu q^\nu k^\sigma - q^\alpha k^\mu k^\nu + \\
+ g^{\mu\nu}(k^\sigma k^2 + k^\sigma q^2) - g^{\mu\sigma}(q^\nu k^2 - k^\nu (kq)) + g^{\nu\sigma}(q^\mu (qk) - k^\mu q^2) \Big]
\end{aligned}
\tag{3.46}
$$

Next we proceed with box Feynman diagrams involving tensor mesons.

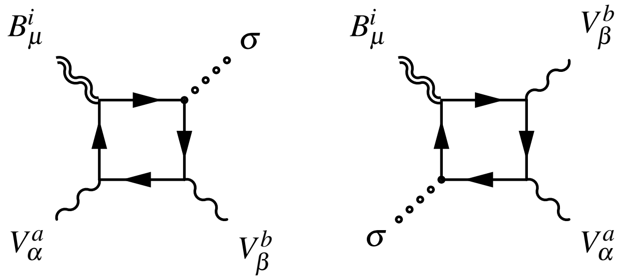


Fig. 3.25 Quantum interaction between B, V and σ meson.

$$
\begin{aligned}
\mathscr{L}_{BVV\sigma} = i N_c \frac{g_B}{|k|} g_V^2 g_\sigma \mathrm{Tr}(\tau^i \tau^a \tau^b) i I_0 4 \varepsilon^{\alpha\beta\mu k} = \\
= -4 I_0 N_c \frac{g_R}{|k|} g_V^2 g_\sigma \mathrm{Tr}(\tau^i \tau^a \tau^b) \varepsilon^{\alpha\beta\mu k}
\end{aligned}
\tag{3.47}
$$

Following a similar argument

$$
\mathscr{L}_{BAA\sigma} = -2 \frac{g_B}{|k|} g_\sigma g_A^2 N_c \mathrm{Tr}[\tau^i \tau^a \tau^b] I_0 \varepsilon^{\alpha\mu\nu k} \tag{3.48}
$$

$$
\mathscr{L}_{RVV\sigma} = 4 I_0 N_c \frac{g_R}{|k|} g_V^2 g_\sigma \mathrm{Tr}(\tau^i \tau^j \tau^k)(k^\beta g^{\mu\alpha} - k^\alpha g^{\mu\beta}) \tag{3.49}
$$

$$\mathscr{L}_{RAA\sigma} = 4I_0 N_c \frac{g_R}{|k|} g_A^2 g_\sigma \mathrm{Tr}(\tau^i \tau^j \tau^k)(k^\beta g^{\mu\alpha} - k^\alpha g^{\mu\beta}) \tag{3.50}$$

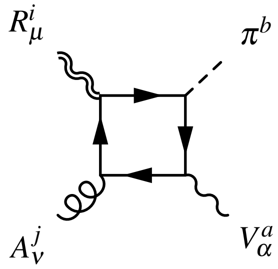


Fig. 3.26 Quantum interaction between R, V, A and π meson.

$$\begin{aligned}\mathscr{L}_{RVPA} &= iN_c \frac{g_R}{|k|} g_V g_\pi g_A \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}(\sigma_{\mu k} \hat{p} \gamma_\nu \hat{p} \gamma_\alpha \hat{p} \hat{p})}{p^8} = \\ &= 2N_c \frac{g_R}{|k|} g_V g_\pi g_A I_0 (k^\alpha g^{\mu\nu} - k^\nu g^{\alpha\mu}) \mathrm{Tr}(\tau^i \tau^j \tau^a \tau^b)\end{aligned} \tag{3.51}$$

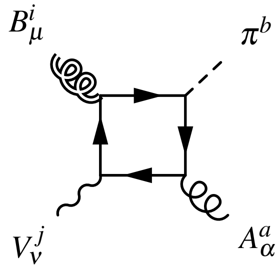


Fig. 3.27 Quantum interaction between B, A, π and V meson.

$$\begin{aligned}\mathscr{L}_{BVA\pi} &= -\frac{g_B}{|k|} g_V g_A g_\pi N_c \mathrm{Tr}(\tau^i \tau^j \tau^a \tau^b) \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu k} \gamma^5 \hat{p} \gamma^\nu \hat{p} \gamma^\alpha \gamma^5 \hat{p} \gamma^5 \hat{p}\right]}{p^8} = \\ &= -2i \frac{g_B}{|k|} g_V g_A g_\pi N_c \mathrm{Tr}(\tau^i \tau^j \tau^a \tau^b) I_0 \varepsilon^{\alpha\mu\nu k}\end{aligned} \tag{3.52}$$

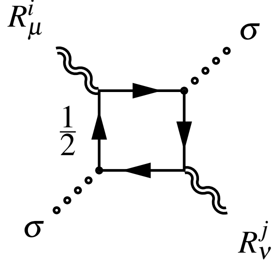


Fig. 3.28 Quantum contribution diagram to the mass of the R meson.

$$
\begin{aligned}
\mathscr{L}_{RR\sigma\sigma} &= -\frac{1}{2} i N_c \frac{g_R^2}{k^2} g_\sigma^2 \mathrm{Tr}(\tau^i \tau^j) \int \frac{d^4 p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu k} \hat{p} \hat{p} \sigma_{\nu q} \hat{p} \hat{p}\right]}{p^8} = \\
&= 2 N_c \frac{g_R^2}{k^2} g_\sigma^2 \mathrm{Tr}(\tau^i \tau^j) \left[k^2 g^{\mu\nu} - k^\mu k^\nu\right] I_0
\end{aligned}
\tag{3.53}
$$

From here we can easily deduce the result from the diagram containing two B mesons and two σ.

$$
\mathscr{L}_{BB\sigma\sigma} = 2 N_c \frac{g_B^2}{k^2} g_\sigma^2 \mathrm{Tr}(\tau^i \tau^j) \left[k^2 g^{\mu\nu} - k^\mu k^\nu\right] I_0 \tag{3.54}
$$

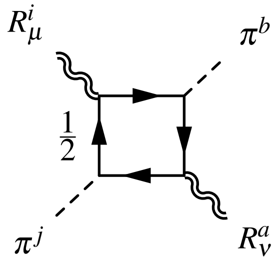


Fig. 3.29 Quantum interaction diagram between R meson and π.

$$
\begin{aligned}
\mathscr{L}_{RR\pi\pi} &= -\frac{1}{2} i \frac{g_R^2}{k^2} g_\pi^2 \mathrm{Tr}(\tau^i \tau^j \tau^a \tau^b) N_c \int \frac{d^4 p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu k} \hat{p} \gamma^5 \hat{p} \sigma_{\nu k} \hat{p} \gamma^5 \hat{p}\right]}{p^8} = \\
&= 2 \frac{g_R^2}{k^2} g_\pi^2 \mathrm{Tr}(\tau^i \tau^j \tau^a \tau^b) N_c I_0 (k^2 g^{\mu\nu} - k^\mu j^\nu)
\end{aligned}
\tag{3.55}
$$

In a similar manner we get the interaction between a B meson and π meson:

$$
\mathscr{L}_{BB\pi\pi} = 2 \frac{g_B^2}{k^2} g_\pi^2 N_c I_0 \mathrm{Tr}(\tau^i \tau^j \tau^a \tau^b) \left(k^\mu k^\nu - k^2 g^{\mu\nu}\right) \tag{3.56}
$$

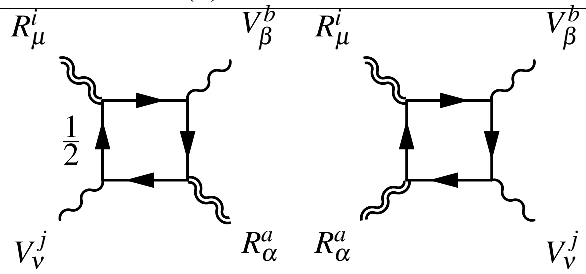


Fig. 3.30 Quantum interaction diagram between R meson and V.

$$\mathscr{L}_{RRVV} = \frac{1}{2} i \frac{g_R^2}{k^2} g_V^2 \mathrm{Tr}(\tau^i \tau^j \tau^a \tau^b) N_c \int \frac{d^4 p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu k}\hat{p}\gamma^\nu \hat{p}\sigma_{\alpha k}\hat{p}\gamma^\beta \hat{p}\right]}{p^8} + \\ + i \frac{g_R^2}{k^2} g_V^2 \mathrm{Tr}(\tau^i \tau^a \tau^j \tau^b) N_c \int \frac{d^4 p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu k}\hat{p}\sigma_{\alpha k}\hat{p}\gamma_n u \hat{p}\gamma_\beta \hat{p}\right]}{p^8} = \\ = \frac{2}{3} \frac{g_R^2}{k^2} g_V^2 N_c I_o \mathrm{Tr}\left[\tau^i [\tau^a, \tau^j] \tau^b\right] T, \tag{3.57}$$

$$T = k^\alpha k^\beta g^{\mu\nu} - k^2 g^{\alpha\beta} g^{\mu\nu} - k^\alpha k^\mu g^{\beta\nu} + k^\beta k^\mu g^{\alpha\nu} + \\ + k^\alpha k^\nu g^{\beta\mu} - 2k^\beta k^\nu g^{\alpha\mu} + k^\mu k^\nu g^{\alpha\beta} - k^2 g^{\alpha\nu} g^{\beta\mu} + k^2 g^{\alpha\mu} g^{\beta\nu} \tag{3.58}$$

Following similar calculations we can obtain the Lagrangian for the interaction between the B and V meson and the B and A meson:

$$\mathscr{L}_{BBVV} = \frac{2}{3} \frac{g_B^2}{k^2} g_V^2 N_c I_0 \mathrm{Tr}\left[\tau^i [\tau^a, \tau^j] \tau^b\right] T \tag{3.59}$$

$$\mathscr{L}_{BBAA} = \frac{2}{3} \frac{g_B^2}{k^2} g_A^2 N_c I_0 \mathrm{Tr}\left[\tau^i \{\tau^a, \tau^j\} \tau^b\right] T \tag{3.60}$$

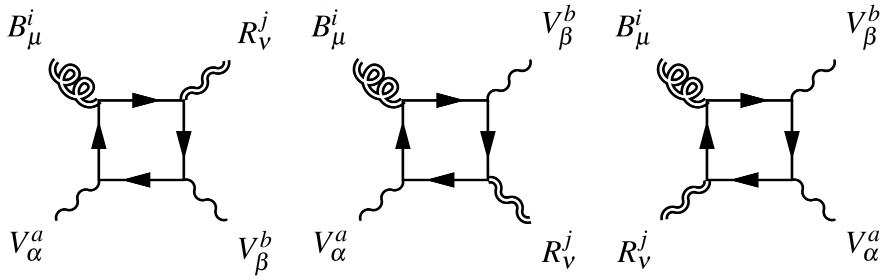


Fig. 3.31 Quantum interaction diagram between B, R and V meson.

$$
\begin{aligned}
\mathscr{L}_{RBVV} = & -g_V^2 \frac{g_R g_B}{k^2} N_c \mathrm{Tr}[\tau^i \tau^a \tau^b \tau^j] \int \frac{d^4 p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu k} \gamma^5 \hat{p} \gamma^\alpha \hat{p} \gamma^\beta \hat{p} \sigma_{\nu k} \hat{p}\right]}{p^8} - \\
& -g_V^2 \frac{g_R g_B}{k^2} N_c \mathrm{Tr}[\tau^i \tau^a \tau^j \tau^b] \int \frac{d^4 p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu k} \gamma^5 \hat{p} \gamma^\alpha \hat{p} \sigma_{\nu k} \hat{p} \gamma^\beta \hat{p}\right]}{p^8} - \\
& -g_V^2 \frac{g_R g_B}{k^2} N_c \mathrm{Tr}[\tau^i \tau^j \tau^a \tau^b] \int \frac{d^4 p}{(2\pi)^4} \frac{\mathrm{Tr}\left[\sigma_{\mu k} \gamma^5 \hat{p} \sigma_{\nu k} \hat{p} \gamma^\alpha \hat{p} \gamma^\beta \hat{p}\right]}{p^8} - \\
= & \frac{2}{3} g_V^2 \frac{g_R g_B}{k^2} N_c I_0 \mathrm{Tr}[\{\tau^i \tau^j\} \tau^a \tau^b] (2k^\alpha \varepsilon^{\beta\mu\nu k} + k^\mu \varepsilon^{\alpha\beta\nu k} - k^\nu \varepsilon^{\alpha\beta\mu k} + k^2 \varepsilon^{\alpha\beta\mu\nu}) - \\
& -\frac{4}{3} g_V^2 \frac{g_R g_B}{k^2} N_c I_0 \mathrm{Tr}[\tau^i \tau^a \tau^j \tau^b] (k^\alpha \varepsilon^{\beta\mu\nu k} + k^\beta \varepsilon^{\alpha\mu\nu k})
\end{aligned}
\tag{3.61}
$$

Following the same diagram with B, R and A we get

$$
\begin{aligned}
\mathscr{L}_{RBAA} = & \frac{2}{3} g_A^2 \frac{g_R g_B}{k^2} N_c I_0 \mathrm{Tr}[\{\tau^i \tau^j\} \tau^a \tau^b] (2k^\alpha \varepsilon^{\beta\mu\nu k} + k^\mu \varepsilon^{\alpha\beta\nu k} - k^\nu \varepsilon^{\alpha\beta\mu k} + k^2 \varepsilon^{\alpha\beta\mu\nu}) + \\
& + \frac{4}{3} g_A^2 \frac{g_R g_B}{k^2} N_c I_0 \mathrm{Tr}[\tau^i \tau^a \tau^j \tau^b] (k^\alpha \varepsilon^{\beta\mu\nu k} + k^\beta \varepsilon^{\alpha\mu\nu k})
\end{aligned}
\tag{3.62}
$$

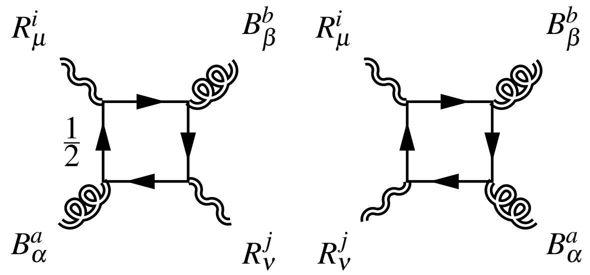


Fig. 3.32 Quantum interaction diagram between R meson and B.

$$
\begin{aligned}
\mathscr{L}_{RRBB} = & -\frac{2}{3} \frac{g_R^2 g_B^2}{k^2} N_c I_0 \mathrm{Tr}[\tau^i \tau^a \tau^j \tau^b] \left(k^\alpha k^\beta - k^2 g^{\alpha\beta}\right) \left(k^\mu k^\nu - k^2 g^{\mu\nu}\right) + \\
& + \frac{4}{3} \frac{g_R^2 g_B^2}{k^2} N_c I_0 \mathrm{Tr}[\tau^i \tau^j \tau^a \tau^b] \left(k^\alpha k^\mu - k^2 g^{\alpha\mu}\right) \left(k^\beta k^\nu - k^2 g^{\beta\nu}\right)
\end{aligned}
\tag{3.63}
$$

And the quantum contributions to the self energy of the tensor mesons are given by

$$
\mathscr{L}_{RRRR} = \frac{1}{3} \frac{g_R^4}{k^4} \mathrm{Tr}[\tau^i \tau^j \tau^a \tau^b] N_c \left(k^\alpha k^\mu - k^2 g^{\alpha\mu}\right) \left(k^\beta k^\nu - k^2 g^{\beta\nu}\right) \tag{3.64}
$$

$$\mathscr{L}_{BBBB} = \frac{1}{3}\frac{g_B^4}{k^4}\mathrm{Tr}[\tau^i\tau^j\tau^a\tau^b]N_c\left(k^\alpha k^\mu - k^2 g^{\alpha\mu}\right)\left(k^\beta k^\nu - k^2 g^{\beta\nu}\right) \tag{3.65}$$

3.5 Spin-1 Mesons Mass Relation

After the full analysis of the non-zero Feynman diagrams in the $SU(2)$ model we can account for the masses of the mesons as well as the description of the chiral symmetry breaking. The free Lagrangian of the model is

$$\begin{aligned}\mathscr{L}_0 = &\frac{1}{2}\left(\partial_\mu\sigma\right)^2 - \frac{\mu_\sigma^2}{2}\sigma^2 - \frac{1}{4}\vec{V}_{\mu\nu}^2 + \frac{\mu_V^2}{2}\vec{V}_\mu^2 - \frac{1}{4}\vec{R}_{\mu\nu}^2 + \frac{\mu_R^2}{2}\vec{R}_\mu^2 + \\ &+\frac{1}{2}\left(\partial_\mu\vec{\pi}\right)^2 - \frac{\mu_\pi^2}{2}\vec{\pi}^2 - \frac{1}{4}\vec{A}_{\mu\nu}^2 + \frac{\mu_A^2}{2}\vec{A}_\mu^2 - \frac{1}{4}\vec{B}_{\mu\nu}^2 + \frac{\mu_B^2}{2}\vec{B}_\mu^2\end{aligned} \tag{3.66}$$

At quantum level due to radiation corrections we obtain an additional term occurs for the self interaction of scalar mesons with zero external momenta. From formula 3.23 we get that the effective potential of the scalar fields resembels a Higgs field

$$V[S] = \frac{\mu_\sigma^2}{2}\sigma^2 + \frac{g^2}{2}\sigma^4 \tag{3.67}$$

That is why we say that sigma mesons in this model act as the Higgs bosons in the Standard Model. In order for the chiral symmetry to be spontaneously broken we insist that $G_0 > 1/(8N_C I_2)$ and, hence, negative μ_σ^2. The symmetry breaking leads to the generation of nonzero constituent quark mass $m = -g\langle\sigma\rangle$, where $\langle S\rangle^2 = -\mu_\sigma^2/(2g^2)$. Therefore, nonzero vacuum expectation values of the S field leads to quantum contributions to the masses of the other meson states

$$\Delta\mathscr{L}_{\mu^2} = -g_\pi^2\langle\sigma\rangle^2\vec{\pi}^2 + 2g_A^2\langle\sigma\rangle^2\vec{A}_\mu^2 - g_R^2\langle\sigma\rangle^2\vec{R}_\mu^2 + g_B^2\langle\sigma\rangle^2\vec{B}_\mu^2. \tag{3.68}$$

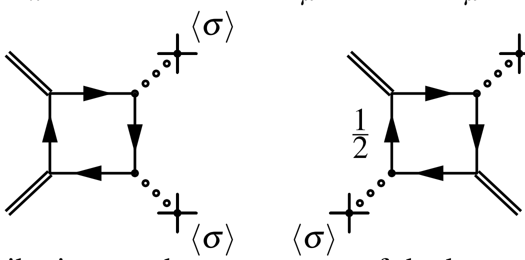


Fig. 3.33 Quantum contributions to the mass terms of the boson states after spontaneous symmetry breaking. Here the double line is corresponding to any of the $\vec{P}$, $\vec{A}_\mu$, $\vec{R}_\mu$, or $\vec{B}_\mu$ bosons.

the contribution of the first term in 3.68, $-g_P^2\langle S\rangle^2 = \mu^2/2$, leads to compensation of the mass term $\mu_P^2/2$ in 3.66 and to masslessness of the pseudoscalar state. An absence of the mass term for the pseudoscalar boson, is a direct consequence of the Goldstone theorem [38]. Thus, as a result of spontaneous chiral SU(2) symmetry breaking, we obtain a triplet of massless pseudoscalar particles, which correspond to physical pion states.

Since the π and the A states are not pure we ahve mixing between them, as well as between the V and R states.

$$\Delta\mathscr{L}_{\text{mixing}} = 2g_\pi g_A \frac{\langle\sigma\rangle}{g} \left(\partial_\mu \vec{\pi}\right) \vec{A}_\mu + g_V g_R \frac{\langle\sigma\rangle}{g} \left(\sqrt{-\partial^2}\vec{V}_\mu\right) \vec{R}_\mu . \tag{3.69}$$

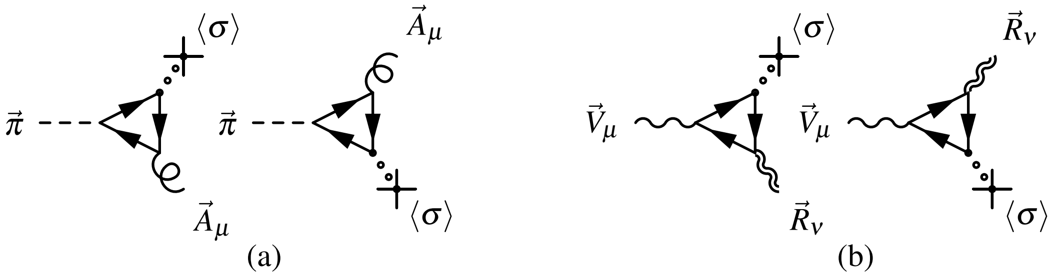


Fig. 3.34 Quantum contributions to the mixing between different boson states after spontaneous symmetry breaking.

In order to diagonalize the term governing the mixing between the π and A state we introduce a new vector field $(\vec{a}_1)_\mu = \vec{A}_\mu - \sqrt{6}m\,(\partial_\mu \vec{P})/m_{a_1}^2$, where $m_{a_1} = \sqrt{\mu_A^2 + 6m^2}$ is its mass. Then the physical pion state is defined as normilized $\vec{\pi}$ state:

$$\vec{\pi}_{\text{phys}} = Z^{-1/2}\,\vec{\pi} = \sqrt{1 - \frac{6m^2}{m_{a_1}^2}}\,\vec{\pi}. \tag{3.70}$$

The mixing between the V and R field is diagonalized by introducing two physical states with the same quantum numbers

$$\begin{aligned} \vec{\rho}_\mu(q^2) &= \cos\theta(q^2)\vec{V}_\mu + \sin\theta(q^2)\vec{R}_\mu, \\ \vec{\rho}\,'_\mu(q^2) &= -\sin\theta(q^2)\vec{V}_\mu + \cos\theta(q^2)\vec{R}_\mu. \end{aligned} \tag{3.71}$$

Collecting all necessary terms with $\vec{V}_\mu$ and $\vec{R}_\mu$ fields from 3.66, 3.68 and 3.69 we have

$$\begin{aligned}\mathscr{L}_{VR} &= -\frac{1}{2}\left(\vec{V}_\mu\ \vec{R}_\mu\right)\begin{pmatrix} q^2-\mu_V^2 & \sqrt{18m^2q^2} \\ \sqrt{18m^2q^2} & q^2-\mu_T^2+6m^2\end{pmatrix}\begin{pmatrix}\vec{V}_\mu \\ \vec{R}_\mu\end{pmatrix} = \\ &= -\frac{1}{2}\left(\vec{\rho}_\mu\ \vec{\rho}\,'_\mu\right)\begin{pmatrix} q^2-m_\rho^2(q^2) & 0 \\ 0 & q^2-m_{\rho'}^2(q^2)\end{pmatrix}\begin{pmatrix}\vec{\rho}_\mu \\ \vec{\rho}\,'_\mu\end{pmatrix}.\end{aligned} \tag{3.72}$$

Here the functions

$$m_\rho^2(q^2) = \Sigma - \sqrt{18m^2q^2+\Delta^2} \quad \text{and} \quad m_{\rho'}^2(q^2) = \Sigma + \sqrt{18m^2q^2+\Delta^2}\,, \tag{3.73}$$

where $\Sigma = (\mu_T^2+\mu_V^2-6m^2)/2$ and $\Delta = (\mu_T^2-\mu_V^2-6m^2)/2$, play the role of the mass operators of the physical particles ρ and ρ', respectively. In the general case the mixing angle

$$\theta(q^2) = \frac{1}{2}\arctan\frac{\sqrt{18m^2q^2}}{\Delta}\,, \tag{3.74}$$

is a function of the square of the momentum q^2, unless condition $\Delta = 0$ is not fulfilled.

The meson state B is a pure state which allows us to eliminate μ_T from the equations

$$m_{b_1} = \sqrt{\mu_T^2+6m^2} \tag{3.75}$$

and therefore we obtain an equation for the constituent quark mass m:

$$m^2 = \frac{2\left(m_{\rho'}^2+m_\rho^2\right)-m_{b_1}^2-\sqrt{\left[2\left(m_{\rho'}^2+m_\rho^2\right)-3m_{b_1}^2\right]^2+8m_{\rho'}^2m_\rho^2}}{24} = (162\pm 7\ \text{MeV})^2\,, \tag{3.76}$$

and $\mu_V^2 = m_{\rho'}^2+m_\rho^2-m_{b_1}^2-6m^2 = (1039\pm 33\ \text{MeV})^2$, using the physical boson masses [39]: $m_\rho^{\text{PDG}} = 775.26\pm 0.25$ MeV, $m_{b_1}^{\text{PDG}} = 1229.5\pm 3.2$ MeV and $m_{\rho'}^{\text{PDG}} = 1465\pm 25$ MeV. For such parameters the mixing angles (3.74) at different masses, $\theta(m_\rho^2) = 41.8^\circ\pm 2.8^\circ$ and $\theta(m_{\rho'}^2) = 43.3^\circ\pm 1.5^\circ$, turn out to be almost constant within experimental errors and close to the maximal mixing angle of 45°. Thus, we can accept the hypothesis of maximal mixing for the vector mesons with good accuracy.

This gives novel relation among boson masses [23], namely,

$$R \equiv \frac{2m_{\rho'}^2-m_{\rho'}m_\rho+2m_\rho^2}{3m_{b_1}^2} = 1\,. \tag{3.77}$$

It should be compared with the experimental value $R_{I=1}^{\exp} = 0.96 \pm 0.03$. The hypothesis works even better in U(1) case for isoscalar $I = 0$ mesons. So, the vector mesons, $\omega(782)$ and $\omega'(1420)$, and the axial-vector meson $h_1(1170)$ are consisting of light quarks in singlet state. Using their accepted mass values [39]: $m_{\omega}^{\mathrm{PDG}} = 782.65 \pm 0.12$ MeV, $m_{h_1}^{\mathrm{PDG}} = 1170 \pm 20$ MeV, $m_{\omega'}^{\mathrm{PDG}} = 1425 \pm 25$ MeV, we find that the ratio $R_{I=0}^{\exp} = 1.02 \pm 0.07$ also agrees with unity within the experimental uncertainty. Moreover, the relation (3.77) predicts exactly the mass of the recently discovered by the BESIII Collaboration [40] axial-vector strangeonium meson $h_1(s\bar{s})$: $m_{h_1(s\bar{s})} = 1415.5 \pm 13.4$ MeV, where the known masses $m_{\phi}^{\mathrm{PDG}} = 1019.461 \pm 0.016$ MeV and $m_{\phi'}^{\mathrm{PDG}} = 1680 \pm 20$ MeV have been used. It should be compared with the experimental value $m_{h_1(1380)}^{\exp} = 1423.2 \pm 2.1 \pm 7.3$ MeV measured by the BESIII Collaboration [40] and the average Particle Data Group value [39] $m_{h_1(1380)}^{\mathrm{PDG}} = 1407 \pm 12$ MeV.

We have already considered states consisting of u and d quarks. Now we discuss the applicability of the mass relation 3.77 for $U(1)$ symmetry. The model with a massless quark is completely analogous to the $SU(2)$ theory we already developed. Now we consider singlet states with quark-antiquark content. In the case uf singlet states with u and d quarks we get $\omega(1420)$ mesons and axial-vector mesons h_1 and f_1. The masses of these mesons form the relation

$$R_{I=0} = \frac{2m_{\omega}^2 - m_{\omega}m_{\omega'} + 2m_{\omega'}^2}{3m_{h_1}^2} = 1 \tag{3.78}$$

Taking into account the masses of the mesons we get our prediction $R_{I+0}^{\exp} = 1.02 \pm 0.07$ which is consistent with the value of 1. Using analogous mass relation to 3.77 we can predict the mass of $h_1(s\bar{s})$ with hiddent strangeness which is absent from the main table in PDG. We can predict that

$$m_{h_1(s\bar{s})}^2 = \frac{2m_{\phi} - m_{\phi}m_{\phi'} + 2m_{\phi'}^2}{3} = (1415.5 \pm 13.4\mathrm{MeV})^2 \tag{3.79}$$

Before that prediction in [23] two collaborations have detected this meson - LASS [41] and Chrystal Barrel [42] with prediction 1380 ± 20MeV and 1440 ± 60MeV correspondingly. BESIII collaboration published a more accurate result for the mass of this meson: $1412 \pm 4 \pm 8$MeV [43]. A more accurate recent result has been obtained by the same collaboration $1423.2 \pm 2.1 pm 7.3$MeV [40]. Combining all existing results we get

$$m_{h_1}^{\exp} = 1415.7 \pm 5.5\mathrm{MeV} \tag{3.80}$$

which is in a perfect agreement (0.05σ) with our prediction. This way we can use our mass relation for predicting the masses of new mesons. In $U(1)$ the constituent quark masses are

$$m_{I=0}^2 = \frac{2\left(m_\omega^2 + m_{\omega'}^2\right) - m_{h_1}^2 - \sqrt{\left[2\left(m_{\omega'}^2 + m_\omega^2\right) - 3m_{h_1}^2\right]^2 + 8m_{\omega'}^2 m_\omega^2}}{24} = (151 \pm 6\ \text{MeV})^2\,, \tag{3.81}$$

and

$$m_{s\bar{s}}^2 = \frac{2\left(m_\phi^2 + m_{\phi'}^2\right) - m_{h_1(s\bar{s})}^2 - \sqrt{\left[2\left(m_{\phi'}^2 + m_\phi^2\right) - 3m_{h_1(s\bar{s})}^2\right]^2 + 8m_{\phi'}^2 m_\phi^2}}{24} = (156 \pm 5\ \text{MeV})^2\,, \tag{3.82}$$

It is interesting to know that all values for the constituent mass are in good agreement within the experimental error. This proves the universality of the mechanism of the pentaneous breaking of chiral symmetry, which is independent from the Higgs mechanism.

Chapter 4

U(1) Nambu-Jona-Lasinio Model with a Massive Quark

In a previous chapter we discussed the implications of a $U(1)$ NJL model with a massless quark, starting from an effective Lagrangian including only interactions between mesons and a fermion. Calculating quantum corrections to the Lagrangian we obtain interaction and kinetic terms for each field. We managed to derive a new mass relation between the mesons predicted by the model. Now we want to further investigate the boundaries of the already existing model and predictions, by including a massive quark. Then we check if any additional corrections will be introduced in the already established relation between the meson masses.

4.1 Relevant Self-interactions

We start with the following Lagrangian

$$\mathscr{L} = \bar{\Psi}(\hat{q} - m_0)\Psi + g_S\bar{\Psi}\Psi S + g_V\bar{\Psi}\gamma_\mu\Psi V_\mu - ig_R\bar{\Psi}\frac{\sigma_{\mu\nu}q^\mu}{|q|}\Psi R_\nu + g_B\bar{\Psi}\frac{\sigma_{\mu\nu}q_\mu\gamma^5}{|q|}\Psi B_\nu \quad (4.1)$$

The arising self-interactions are shown on Figure 4.1:

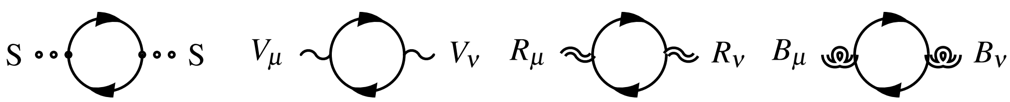


Fig. 4.1 Feynman diagrams self-interactions for the arising spin-1 excitations.

We calculate the contribution of each diagram in turn. By using the Feynman rules the first diagram from Figure 4.1 becomes

$$\Pi^S(q^2) = ig_S^2 N_c \int \frac{d^4p}{(2\pi)^4} \frac{Tr\,[(\hat{p}-\hat{q}+m_0)(\hat{p}+m_0)]}{(p^2-m_0^2)^2}$$
$$\left[1+\frac{2(pq)-q^2}{p^2-m_0^2}+\frac{4(pq)^2}{(p^2-m_0^2)^2}+...\right] = 4g_S^2 N_c I_2 + 2g_S^2 N_c q^2 I_0 - 8m_0^2 g_S^2 N_c I_0 \tag{4.2}$$

From here we define $\mu_S = \frac{g_S^2}{G_0} - 4g_S^2 N_c I_2 + 8m_0^2 g_S^2 N_c I_0$

The second diagram from Figure 4.1 has the following contribution:

$$\Pi^V_{\mu\nu}(q^2) = ig_V^2 N_c$$
$$\int \frac{d^4p}{(2\pi)^4} \frac{Tr\,[\gamma^\mu(\hat{p}-\hat{q}+m_0)\gamma^\nu(\hat{p}+m_0)]}{(p^2-m_0^2)^2} \left[1+\frac{2(pq)-q^2}{p^2-m_0^2}+\frac{4(pq)^2}{(p^2-m_0^2)^2}+...\right] \tag{4.3}$$

We use that

$$Tr\,[\gamma^\mu(\hat{p}-\hat{q}+m_0)\gamma^\nu(\hat{p}+m_0)] = 2p^\mu p^\nu - p^2 g^{\mu\nu} - p^\mu q^\nu - p^\nu q^\mu + (pq)g^{\mu\nu} + m_0^2 g^{\mu\nu}$$

to get the final result

$$\Pi^V_{\mu\nu}(q^2) = -2N_c g_V^2 I_2 g^{\mu\nu} - 2N_c g_V^2 m_0^2 g^{\mu\nu} I_0 + \frac{4}{3} I_0 N_c g_V^2 (q^\mu q^\nu - q^2 g^{\mu\nu}) \tag{4.4}$$

From here we define $\mu_V^2 = \frac{g_V^2}{G_V} - 2N_c g_V^2 I_2 - \frac{3}{2}m_0^2$

The third diagram from Figure 4.1 is equal to

$$\Pi^R_{\mu\nu}(q^2) = i\frac{g_R^2}{q^2} N_c$$
$$\int \frac{d^4p}{(2\pi)^4} \frac{Tr\left[\sigma_{\mu q}(\hat{p}-\hat{q}+m_0)\sigma_{\nu q}(\hat{p}+m_0)\right]}{(p^2-m_0^2)^2} \left[1+\frac{2(pq)-q^2}{p^2-m_0^2}+\frac{4(pq)^2}{(p^2-m_0^2)^2}+...\right] \tag{4.5}$$

We use that

$$T_1 = Tr\left[\sigma_{\mu q}(\hat{p}-\hat{q}+m_0)\sigma_{\nu q}(\hat{p}+m_0)\right] = -4(2g^{\mu\nu}(pq)^2 - m_0^2 q^2 g^{\mu\nu} - p^2 q^2 g^{\mu\nu} -$$

$$q^2(pq)g^{\mu\nu} + m_0^2 q^\mu q^\nu + p^2 q^\mu q^\nu + 2q^2 p^\mu p^\nu - 2p^\nu q^\mu (pq) - 2p^\mu q^\nu (pq) + q^\mu q^\nu (pq))$$

to get

$$\Pi^R_{\mu\nu}(q^2) = \frac{2}{3} g_R^2 N_c I_0 \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) + 4 g_R^2 N_c I_0 \frac{m_0^2}{q^2}\left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) \tag{4.6}$$

and we define $\mu_R^2 = \frac{g_T^2}{G_T}$

The final diagram from Figure 4.1 is

$$-i\frac{g_B^2}{q^2}N_c \int \frac{d^4p}{(2\pi)^4}\frac{T_2}{(p^2-m_0^2)^2}\left[1+\frac{2(pq)}{p^2-m_0^2}+\frac{4(pq)^2}{(p^2-m_0^2)^2}+...\right]$$

$$T_2 = Tr\left[\sigma_{\mu q}\gamma^5(\hat{p}-\hat{q}+m_0)\sigma_{\nu q}\gamma^5(\hat{p}+m_0)\right] =$$

$$4(2g^{\mu\nu}(pq)^2 + m_0^2 q^2 g^{\mu\nu} - p^2 q^2 g^{\mu\nu} - q^2(pq)g^{\mu\nu} - m_0^2 q^\mu q^\nu +$$

$$+p^2 q^\mu q^\nu + 2q^2 p^\mu p^\nu - 2p^\nu q^\mu (pq) - 2p^\mu q^\nu (pq) + q^\mu q^\nu (pq)) \Rightarrow$$

$$\Pi^B_{\mu\nu}(q^2) = \frac{2}{3} g_B^2 N_c I_0 \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) - 4 g_B^2 N_c I_0 \frac{m_0^2}{q^2}\left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) \tag{4.7}$$

We define $\mu_B^2 = \frac{g_T^2}{G_T} = \mu_R^2 \equiv \mu_T^2$. Using the normalization condition we can write the relation between all interaction constants:

$$3g^2 = 3g_S^2 = 2g_V^2 = g_R^2 = g_B^2 = \frac{3}{2N_c I_0} \tag{4.8}$$

We can write the free Lagrangian of the boson fields as

$$\begin{aligned}\mathscr{L}_0 = \frac{1}{2}\left(\partial_\mu S\right)^2 - \frac{\mu_S^2}{2}S^2 - \frac{1}{4}V_{\mu\nu}^2 + \frac{\mu_V^2}{2}V_\mu^2 - \\ +\frac{1}{4}R_{\mu\nu}^2 + \frac{\mu_T^2 - 6m_0^2}{2}R_\mu^2 - \frac{1}{4}B_{\mu\nu}^2 + \frac{\mu_T^2 + 6m_0^2}{2}B_\mu^2,\end{aligned} \tag{4.9}$$

where $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, $R_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu$ and $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ are the Yang-Mills fields.

4.2 Quantum Contributions to Mass Terms

The constituent quark mass is given by $m = m_0 - g\langle S\rangle$. All further diagrams will be considered after spontaneous symmetry breaking so instead of S we have the vacuum expectation value $\langle S\rangle$.

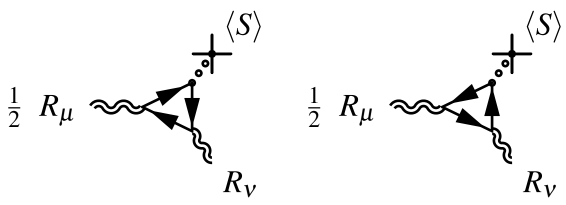


Fig. 4.2 Quantum contribution to the mass of the R_μ meson.

The sum of the diagrams shown on Figure 4.2 give the following contribution

$$
\begin{aligned}
-\frac{1}{2}ig\langle S\rangle\frac{g_R^2}{q^2}N_c\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2-m_0^2)^3}\left[1+2\frac{q(2p-q)}{p^2-m_0^2}+\ldots\right] \\
\Big\{Tr\Big[\sigma_{\mu q}(\hat{p}-\hat{q}+m_0)(\hat{p}-\hat{q}+m_0)\sigma_{\nu q}(\hat{p}+m_0)+ \\
\sigma_{\mu q}(-\hat{p}+m_0)\sigma_{\nu q}(-\hat{p}+\hat{q}+m_0)(-\hat{p}+\hat{q}+m_0)\Big]\Big\} = \\
=4\frac{g_R^2}{q^2}N_cI_0g\langle S\rangle m_0(q^2g_{\mu\nu}-q^\mu q^\nu)
\end{aligned}
\tag{4.10}
$$

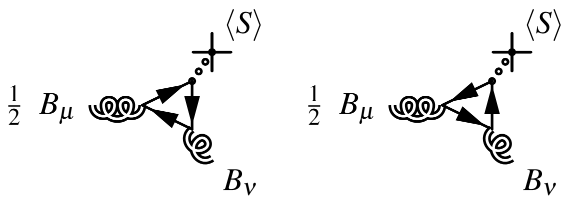


Fig. 4.3 Quantum contribution to the mass of the B_μ meson.

The sum of the diagrams on Figure 4.3 gives

$$\begin{aligned}
&\frac{1}{2}ig\langle S\rangle\frac{g_B^2}{q^2}N_c\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2-m_0^2)^3}\left[1+2\frac{q(2p-q)}{p^2-m_0^2}+...\right]\\
&\Big\{Tr\Big[\sigma_{\mu q}\gamma^5(\hat p-\hat q+m_0)(\hat p-\hat q+m_0)\sigma_{\nu q}\gamma^5(\hat p+m_0)-\\
&-\sigma_{\mu q}\gamma^5(-\hat p+m_0)\sigma_{\nu q}\gamma^5(-\hat p+\hat q+m_0)(-\hat p+\hat q+m_0)\Big]\Big\}=\\
&=-4\frac{g_B^2}{q^2}N_cI_0g\langle S\rangle m_0(q^2g^{\mu\nu}-q^\mu q^\nu)
\end{aligned} \tag{4.11}$$

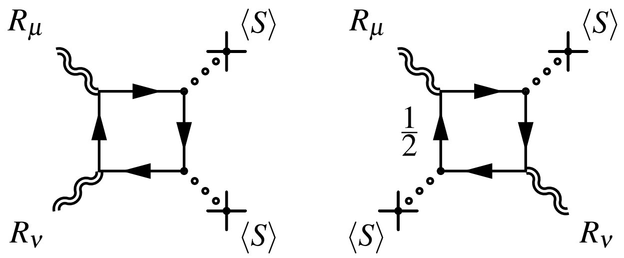


Fig. 4.4 Quantum contribution to the mass of the R_μ meson from a box diagram.

The contribution from the diagrams on Figure 4.4 is

$$\begin{aligned}
&-\frac{1}{2}\frac{g_R^2}{q^2}(ig)^2\langle S\rangle^2N_c\int\frac{d^4p}{(2\pi)^4}\frac{Tr[\sigma_{\mu q}\hat p\hat p\sigma_{\nu q}(\hat p+\hat q)(\hat p+\hat q)]}{p^8}-\\
&-i\frac{g_R^2}{q^2}(ig)^2\langle S\rangle^2N_c\int\frac{d^4p}{(2\pi)^4}\frac{Tr[\sigma_{\mu q}\hat p\sigma_{\nu q}(\hat p+\hat q)(\hat p+\hat q)(\hat p+\hat q)]}{p^8}=\\
&=2\frac{g_R^2}{q^2}g^2\langle S\rangle^2N_cI_0(q^\mu q^\nu-q^2g^{\mu\nu})
\end{aligned} \tag{4.12}$$

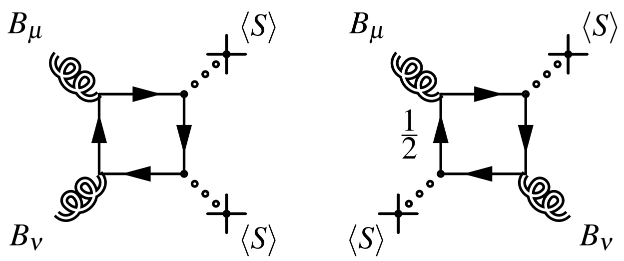


Fig. 4.5 Quantum contribution to the mass of the $B\mu$ meson from a box diagram.

The trace part in the diagrams in Figure 4.5 is the same as in the diagrams in Figure 4.4

$$\frac{1}{2}i\frac{g_B^2}{q^2}(ig)^2\langle S\rangle^2N_c\int...=-2\frac{g_B^2}{q^2}g^2\langle S\rangle^2N_cI_0(q^\mu q^\nu-q^2g^{\mu\nu}) \tag{4.13}$$

The terms which give additional contribution to the masses of the mesons are

$$\Delta\mathscr{L}_\mu = 3(m_0^2 - m^2)R_\mu^2 + 3(m^2 - m_0^2)B_\mu^2 \tag{4.14}$$

4.3 Mixing between the V and the R field

Knowing the quantum numbers of the mesons and Dirac algebra relations we can predict which diagrams would give zero contribution. This way we see that the V and R field are not pure, i.e. there exist particles with equal quantum numbers which are a mix of these two fields. The non-zero mixing diagrams are considered below.

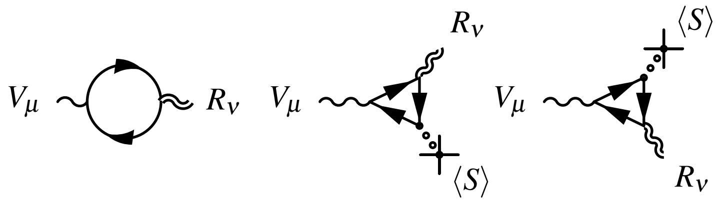


Fig. 4.6 These are the non-zero mixing diagrams between V_μ and R_μ.

For the first diagram on Figure 4.6 we have:

$$\begin{aligned} &\frac{g_R}{|q|} g_V N_c \int \frac{d^4p}{(2\pi)^4} \frac{Tr[\gamma^\mu(\hat{p} - \hat{q} + m_0)\sigma_{\nu q}(\hat{p} + m_0)]}{(p^2 - m_0^2)^2} \\ &\left[1 + \frac{2(pq)}{p^2 - m_0^2} + \frac{4(pq)^2}{(p^2 - m_0^2)^2} + ...\right] = 4g_V \frac{g_R}{|q|} m_0 N_c I_0 (q^\mu q^\nu - q^2 g^{\mu\nu}) \end{aligned} \tag{4.15}$$

From the second and third diagram on Figure 4.6 we obtain:

$$\begin{aligned} ig_V \frac{g_R}{|q|} ig\langle S\rangle N_c \int \frac{d^4p}{(2\pi)^4} &\left[\frac{Tr[\gamma^\mu(-\hat{p} + m_0)(-\hat{p} + m_0)\sigma_{\nu q}(-\hat{p} + \hat{q} + m_0)]}{(p^2 - m_0^2)^3} + \right. \\ &\left. + \frac{Tr[\gamma^\mu(\hat{p} - \hat{q} + m_0)\sigma_{\nu q}(\hat{p} + m_0)(\hat{p} + m_0)}{(p^2 - m_0^2)^3} \right] \left[1 + \frac{2(pq)}{p^2 - m_0^2} + ...\right] = \\ &= 4g_V \frac{g_R}{|q|} N_c I_0 g\langle S\rangle (q^2 g^{\mu\nu} - q^\mu q^\nu) \end{aligned} \tag{4.16}$$

The Lagrangian for the diagram with V_μ and R_μ as two external lines is finally written as

$$\mathscr{L}'_{VR} = -\sqrt{18} m_0 (\sqrt{-\partial^2} V_\mu) R_\mu \tag{4.17}$$

and the Lagrangian for the diagrams with V_μ, R_μ and $\langle S \rangle$ is

$$\mathscr{L}_{VR\langle S\rangle} = \sqrt{18}(m_0 - m)(\sqrt{-\partial^2} V_\mu) R_\mu \tag{4.18}$$

Collecting all terms involving V_μ and R_μ we can write the Lagrangian for these fields in matrix form as

$$\mathscr{L}_{VR} = -\frac{1}{2}\left(V_\mu \; R_\mu\right)\begin{pmatrix} q^2 - \mu_V^2 & \sqrt{18m^2q^2} \\ \sqrt{18m^2q^2} & q^2 - \mu_T^2 + 6m^2 \end{pmatrix}\begin{pmatrix} V_\mu \\ R_\mu \end{pmatrix} \tag{4.19}$$

The eigenvalues of the matrix are the masses of ϕ and ϕ' mesons. Solving the eigenvalue problem we get the following equation we get the masses need to satisfy a quartic equation

$$q^4 + q^2(-\mu_T^2 + 6m^2 - \mu_V^2 - 18m^2) - \mu_V^2(-\mu_T^2 + 6m^2) = 0 \tag{4.20}$$

Using the formulas for the sum and the product of the roots of a quadratic equation (Vieta's formulas) we get

$$m_\phi^2 + m_{\phi'}^2 = \mu_T^2 + \mu_V^2 + 12m^2 \tag{4.21}$$

The maximal mixing condition leads to

$$\mu_V^2 = \mu_T^2 - 6m^2 \tag{4.22}$$

$$m_\phi^2 m_{\phi'}^2 = \mu_V^4 \tag{4.23}$$

The B_μ field can be identified with the h meson which is a pure state. We can deduce its mass from formulas 4.9 and 4.14 and we get $m_h^2 = \mu_T^2 + 6m^2$ and consequently we can eliminate μ_T from the above equations:

$$m_\phi^2 + m_{\phi'}^2 = 2m_h^2 - 6m^2 \tag{4.24}$$

$$m_\phi^2 m_{\phi'}^2 = (m_h^2 - 12m)^2 \tag{4.25}$$

This system of equations leads to the well-known relationship between the masses of all mesons:

$$\frac{2m_\phi^2 - m_\phi m_{\phi'} + 2m_{\phi'}}{3m_h^2} = 1 \tag{4.26}$$

As a conclusion we can observe that the introduction of massive quark in the $U(1)$ NJL model does not introduce any correction to the mass relation between the meson excitations arising within the model.

Chapter 5

Scalar Models and Isotopic Symmetry

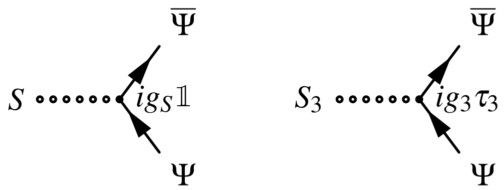
In this chapter we investigate the isotopic symmetry between the light quarks. In the first section we want to check if the isotopic symmetry between the u and d quark can be broken, and therefore show the origin in the difference in the mass between the two quark. The current view for the origin of the different mass in the quarks is that there are different interaction constants between the Higgs boson and the quarks. This is still to be verified both experimentally and on more firm theoretical grounds. But in case the NJL model can give an explanation for the different masses then it would settle a big problem in the modern particle physics. In the first part of the chapter we deal with $U(2)$ symmetry with two quark flavours and three mesons. We linearize the Lagrangian, we calculate all non-zero diagrams, which gives the effective Lagrangian including the interacting potential. By minimizing the interacting potential, after spontaneous symmetry breaking, we will get the values for the masses of the u and d quark. Then we extend our model to $U(3)$ with three quark flavours, where we repeat the whole procedure. All results in this chapter are published in the paper [44].

5.1 U(2) Symmetry

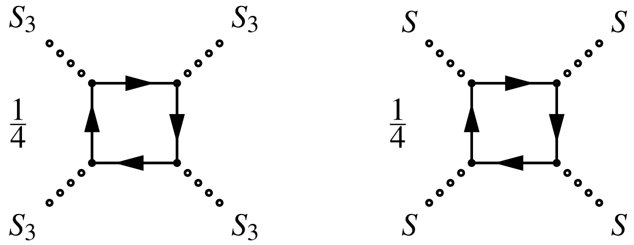
We start with the linearized NJL Lagrangian with a massless quark including only the meson fields, represented by diagonal matrices, since only those fields will give a contribution to the constituent quark masses

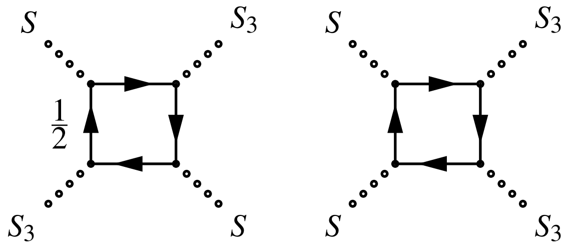
$$\mathscr{L} = \bar{\Psi}\hat{q}\Psi + g\bar{\Psi}\Psi S + g_3\bar{\Psi}\tau_3\Psi S_3 \tag{5.1}$$

The Feynman rules for this theory are



There is no mixing between the two scalars, since the mixing loop diagram is proportional to $\mathrm{Tr}(\tau_3) = 0$. There cannot be diagrams with three external lines as they are proportional to trace of an odd number of gamma matrices for a massless quark case. Therefore, the non-zero diagrams are as follows





$$
\begin{aligned}
\Pi_{SS}(q) = i g_S^2 N_c \mathrm{Tr}[\mathbb{1}] \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[(\hat{p}-m_0)^{-1}(\hat{p}-\hat{q}-m_0)^{-1}\right] = \\
= 8g_S^2 N_c I_2 + 4 g_S^2 N_c I_0 q^2 + \text{fin.terms}
\end{aligned} \tag{5.2}
$$

$$
\begin{aligned}
\Pi_{S_3S_3}(q) = i g_3 S^2 N_c \mathrm{Tr}[\mathbb{1}] \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[(\hat{p}-m_0)^{-1})(\hat{p}-\hat{q}-m_0)^{-1}\right] = \\
= 8g_3^2 N_c I_2 + 4 g_3^2 N_c I_0 q^2
\end{aligned} \tag{5.3}
$$

From the normalization of the divergent integrals we get the relation between the interaction constants:

$$g^2 = g_S^2 = g_3^2 = \frac{1}{4N_c I_0} \text{ and } \mu^2 = \frac{g^2}{G_0} - 8g^2 N_c I_2 \tag{5.4}$$

Then the box diagrams equate to

$$\begin{aligned}\Box_{0000}\frac{i}{4}g^4 N_C \mathrm{Tr}\left[\mathbb{1}\right]\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{Tr}\left[(\hat{p}-m_0)^{-1}(\hat{p}-m_0)^{-1}(\hat{p}-m_0)^{-1}(\hat{p}-m_0)^{-1}\right] = \\ = -2g^4 N_C I_0 + \mathcal{O}(m_0^2) \approx -\frac{1}{2}g^2\end{aligned} \tag{5.5}$$

$$\begin{aligned}\Box_{3333} = \frac{i}{4}g^4 N_C \mathrm{Tr}\left[\tau_3^4\right]\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{Tr}\left[(\hat{p}-m_0)^{-1}(\hat{p}-m_0)^{-1}(\hat{p}-m_0)^{-1}(\hat{p}-m_0)^{-1}\right] = \\ = -2g^4 N_c I_0 = -\frac{1}{2}g^2\end{aligned} \tag{5.6}$$

$$\begin{aligned}\Box_{0303} = \frac{3}{2}ig^4 N_C \mathrm{Tr}\left[\tau_3^2\mathbb{1}^2\right]\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{Tr}\left[(\hat{p}-m_0)^{-1}(\hat{p}-m_0)^{-1}(\hat{p}-m_0)^{-1}(\hat{p}-m_0)^{-1}\right] = \\ = -12g^4 N_c I_0 = -3g^2\end{aligned} \tag{5.7}$$

Summing up all quantum contributions we get the effective interaction potential

$$V[S,S_3] = \frac{\mu^2}{2}(S^2+S_3^2) + \frac{1}{2}g^2(S^4+6S^2S_3^2+S_3^4) \tag{5.8}$$

In order to find extremal points we equate the first derivative with respect to both variables and solve the following system of equations

$$\frac{\partial V}{\partial S} = \mu^2 S + 2g^2 S^3 + 6g^2 S S_3^2 = 0 \tag{5.9}$$

$$\frac{\partial V}{\partial S_3} = \mu^2 S_3 + 2g^2 S_3^3 + 6g^2 S^2 S_3 = 0 \tag{5.10}$$

We also need to investigate the second derivatives of the potential at the points which solve the above system of equations:

$$\begin{aligned}\frac{\partial^2 V}{\partial S^2} &= \mu^2 + 6g^2 S^2 + 6g^2 S_3^2 \\ \frac{\partial^2 V}{\partial S_3^2} &= \mu^2 + 6g^2 S_3^2 + 6g^2 S^2 \\ \frac{\partial^2 V}{\partial S \partial S_3} &= 12g^2 S S_3\end{aligned} \tag{5.11}$$

Let the solution which we are interested in (minimum) is $(\langle S\rangle, \langle S_3\rangle)$. These points need to meet the conditions

$$D[S,S_3] = \frac{\partial^2 V}{\partial S \partial S}(\langle S\rangle, \langle S_3\rangle)\frac{\partial^2 V}{\partial S_3 \partial S_3}(\langle S\rangle, \langle S_3\rangle) - \left[\frac{\partial^2 V}{\partial S \partial S_3}(\langle S\rangle, \langle S_3\rangle)\right]^2 > 0 \tag{5.12}$$

and

$$\frac{\partial^2 V}{\partial S^2}(\langle S\rangle, \langle S_3\rangle) > 0 \tag{5.13}$$

All extremal points which solve the system of the equations 5.9 and 5.10 are the following:

$$A:(0,0),\ \ B:\left(0,-\sqrt{-\frac{\mu^2}{2g^2}}\right)\ \ C:\left(0,\sqrt{-\frac{\mu^2}{2g^2}}\right)$$

$$D:\left(-\sqrt{-\frac{\mu^2}{2g^2}},0\right),\ \ E:\left(\sqrt{-\frac{\mu^2}{2g^2}},0\right)$$

$$F:\left(-\frac{1}{2}\sqrt{-\frac{\mu^2}{2g^2}},-\frac{1}{2}\sqrt{-\frac{\mu^2}{2g^2}}\right),\ \ G:\left(\frac{1}{2}\sqrt{-\frac{\mu^2}{2g^2}},-\frac{1}{2}\sqrt{-\frac{\mu^2}{2g^2}}\right)$$

$$H:\left(-\frac{1}{2}\sqrt{-\frac{\mu^2}{2g^2}},\frac{1}{2}\sqrt{-\frac{\mu^2}{2g^2}}\right),\ \ I:\left(\frac{1}{2}\sqrt{-\frac{\mu^2}{2g^2}},\frac{1}{2}\sqrt{-\frac{\mu^2}{2g^2}}\right)$$

In our particular case

$$D[S,S_3] = \mu^4 + 12\mu^2 g^2 (S^2 + S_3^2) + 36 g^4 (S^2 - S_3^2)^2 \tag{5.14}$$

$$D_A = \mu^4 > 0,\ \ \frac{\partial^2 V}{\partial S^2} = \mu^2 < 0 \tag{5.15}$$

$$D_{BCDE} = 4\mu^4 > 0, \ \frac{\partial^2 V}{\partial S^2} = -\mu^2 > 0 \tag{5.16}$$

$$D_{FGHI} = -2\mu^4 < 0, \ \frac{\partial^2 V}{\partial S^2} = -\frac{\mu^2}{2} > 0 \tag{5.17}$$

Using the classification above 5.12, 5.13 we get that A is a maximum, B,C,D,E are minima, and F,G,H,I are saddle points. We can choose that the u and d quark masses are defined by point D, then

$$-m_{u,d} = g(\langle S \rangle \pm \langle S_3 \rangle) \Rightarrow m_u = m_d = \sqrt{-\frac{\mu^2}{2g^2}} \tag{5.18}$$

We can conclude that the Nambu-Jona-Lasinio model cannot predict the mass splitting between the u and d quark.

5.2 U(3) Symmetry

In this model we want to investigate if isotopic symmetry can be broken, i.e. if we can show that the mass of the u and d quark are different and these two masses are smaller than the s quark mass. Apparently, $U(2)$ is a subgroup of $U(3)$ and if isotopic symmetry is unbroken in $U(2)$ as shown in a previous chapter, then it should remain unbroken in $U(3)$ as well. Here we show physical relations which guarantee the preservation of isotopic symmetry in the present case. The scalar mesons in $U(3)$ have the so-called inverted mass spectrum where the mass of the κ-mesons is smaller than the masses of the $a_0(980)$ and $f_0(980)$ mesons. This experimental observation has not been yet predicted in theory within the Standard Model. Current theories suggest models with 4 quark bound states, or glueballs [45]. Here we suggest a three-flavour quark Nambu-Jona-Lasinio scalar model, where initially we will consider only fields described by diagonal matrices, since only they will give contribution to the constituent quark masses. Then we will extend our model to include all scalar field excitations. Our final goal is to explicitly break the symmetry of the Lagrangian by introducing quark masses, which will be the most realistic scalar model within our theoretical framework.

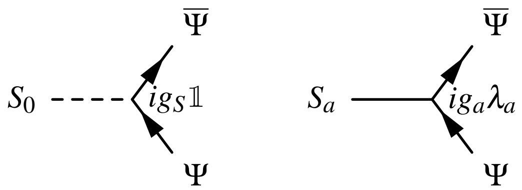
We start with the following Lagrangian:

$$\mathscr{L} = \bar{\Psi}\hat{q}\Psi + \frac{1}{2}G_0(\bar{\Psi}\Psi)^2 + \frac{1}{2}G_0'(\bar{\Psi}\lambda_a\Psi)^2, \tag{5.19}$$

where G_0 and G_0' are different positive constants with dimensions $[\mathrm{mass}]^{-2}$. This is a nonlinear Lagrangian including 4-fermion vertexes. We linearize it by introducing 9 scalar fields in the following manner

$$\mathscr{L} = \bar{\Psi}\hat{q}\Psi + g_S\bar{\Psi}\Psi S_0 + \sum_{a=1}^{8} g_a\bar{\Psi}\lambda_i\Psi S_i - \frac{g_S^2}{2G_0}S_0^2 - \sum_{a=1}^{8}\frac{g_a^2}{2G_0'}S_a^2 \tag{5.20}$$

where $\Psi = (\Psi_1\Psi_2\Psi_3)^T$, $\lambda_a = (\lambda_1,\ldots,\lambda_8)$ are the Gell-Mann matrices, $S_a = (S_1,\ldots,S_8)$ and g_a are dimensionless coupling constants. In this general case the Feynman rules become



Due to the algebra of Gell-Mann matrices we can eliminate most of the diagrams. There cannot exist triangular diagrams because the trace of an odd number of Dirac matrices vanishes. There is no mixing at loop level since $\mathrm{Tr}[\lambda_i\lambda_j] = 2\delta_{ij}$ and the diagrams left are

These diagrams give the following results:

$$ig_S^2 N_c \mathrm{Tr}[\mathbb{1}]_{3\times 3}\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[(\hat{p}-m_0)^{-1}(\hat{p}-m_0)^{-1}\right] = 12g_S^2N_cI_2 + 6g_S^2N_cq^2I_0 \tag{5.21}$$

$$ig_8^2 N_c \mathrm{Tr}[\lambda_a^2]\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[(\hat{p}-m_0)^{-1}(\hat{p}-m_0)^{-1}\right] = 8g_a^2N_cI_2 + 4g_a^2N_cq^2I_0 \tag{5.22}$$

Normalization of the divergent integrals allows us to derive a connection between the three interaction constants, so the theory is dependent on one interaction constant:

$$3g^2 = 3g_S^2 = 2g_a^2 = \frac{1}{2N_cI_0} \tag{5.23}$$

and we also define $\mu^2 = \frac{g^2}{G_0} - 12g^2N_cI_2$ and $\tilde{\mu}^2 = \frac{3}{2}\frac{g^2}{G_0'} - 12g^2N_cI_2$.

As already mentioned - there are no diagrams with three external lines , since the quarks in the model are massless and therefore we are left with trace of an odd number of Dirac matrices. The non-zero diagrams with four external lines (box diagrams) which give contribution to the interacting potential are the following

$$\Box_{iiii} = -\frac{3}{4}g^2 \sum_i S_i^4 \tag{5.24}$$

$$\Box_{00ii} = -3g^2 \sum_i S_0^2 S_i^2 \tag{5.25}$$

$$\Box_{0888} = \sqrt{2}g^2 S_0 S_8^3 \tag{5.26}$$

$$\begin{gathered}\Box_{0iij} = -\frac{3\sqrt{3}}{\sqrt{2}}g^2 \mathrm{Tr}[\lambda_i\lambda_i\lambda_j] S_0 S_i^2 S_j \neq 0 \text{ for} \\ i = 4...7, j = 3 \text{ and } i = 1...7, j = 8 \Rightarrow\end{gathered} \tag{5.27}$$

$$\Box_{0443} = -\frac{3\sqrt{3}}{\sqrt{2}}g^2 S_0 S_4^2 S_3 \tag{5.28}$$

$$\Box_{0553} = -\frac{3\sqrt{3}}{\sqrt{2}}g^2 S_0 S_5^2 S_3 \tag{5.29}$$

$$\Box_{0663} = \frac{3\sqrt{3}}{\sqrt{2}}g^2 S_0 S_6^2 S_3 \tag{5.30}$$

$$\Box_{0773} = \frac{3\sqrt{3}}{\sqrt{2}}g^2 S_0 S_7^2 S_3 \tag{5.31}$$

$$\Box_{0118} = -3\sqrt{2}g^2 S_0 S_1^2 S_8 \tag{5.32}$$

$$\Box_{0228} = -3\sqrt{2}g^2 S_0 S_2^2 S_8 \tag{5.33}$$

$$\Box_{0338} = -3\sqrt{2}g^2 S_0 S_3^2 S_8 \tag{5.34}$$

$$\Box_{0448} = \frac{3\sqrt{2}}{2}g^2 S_0 S_4^2 S_8 \tag{5.35}$$

$$\Box_{0558} = \frac{3\sqrt{2}}{2} g^2 S_0 S_5^2 S_8 \tag{5.36}$$

$$\Box_{0668} = \frac{3\sqrt{2}}{2} g^2 S_0 S_6^2 S_8 \tag{5.37}$$

$$\Box_{0778} = \frac{3\sqrt{2}}{2} g^2 S_0 S_7^2 S_8 \tag{5.38}$$

Next we consider the diagrams $S_0 S_i S_j S_k$, where $i \neq j \neq k$:

$$\Box_{0ijk} = -3\frac{\sqrt{3}}{\sqrt{2}} g^2 \mathrm{Tr}\left[\tau^i \{\tau^j, \tau^k\}\right] S_0 S_i S_j S_k = -6\sqrt{6} g^2 d_{ijk} S_0 S_i S_j S_k \tag{5.39}$$

From the properties of Gell-Mann matrices we have that when the indices are different then

$$d_{ijk} \to d_{146} = \frac{1}{2}; d_{157} = \frac{1}{2}; d_{247} = -\frac{1}{2}; d_{256} = \frac{1}{2} \tag{5.40}$$

and therefore

$$\Box_{0ijk} = -3\sqrt{6} S_0 (S_1 S_4 S_6 + S_1 S_5 S_7 - S_2 S_4 S_7 + S_2 S_5 S_6) \tag{5.41}$$

Finally, we have the case $\Box_{ijkl}$. Due to the algebra of λ matrices such diagrams are non-zero if and only if $i = j$ and $k = l$. Then

$$\Box_{iijj} = -\frac{3}{2} g^2 \sum_{i<j} S_i^2 S_j^2, \tag{5.42}$$

where we have used the fact that $2\mathrm{Tr}[\lambda^i \lambda^i \lambda^j \lambda^j] + \mathrm{Tr}[\lambda^i \lambda^j \lambda^i \lambda^j] = 2$ when $i \neq j$.

Before solving the more general case including all meson excitations we will first consider only those represented by diagonal Gell-Mann matrices since only they will give contribution to the constituent quark masses. We take only the diagonal matrices since if scalars described by offdiagonal matrices obtain VEV then this would mean that charge will disappear in vacuum. Moreover, only diagonal matrices will give contribution to the constituent mass of quarks. Then we arrive at the Lagrangian

$$\mathscr{L} = \bar{\Psi}\hat{q}\Psi + g_S \bar{\Psi}\Psi S_0 + g_3 \bar{\Psi}\lambda_3 \Psi S_3 + g_8 \bar{\Psi}\lambda_8 \Psi S_8 - \frac{g_S^2}{2G_0} S_0^2 - \frac{g_3^2}{2G_0'} S_3^2 - \frac{g_8^2}{2G_0'} S_8^2, \tag{5.43}$$

where the λ_3 and λ_8 matrices are defined as

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{5.44}$$

Substituting back into the original Lagrangian after spontaneous symmetry breaking we get

$$\begin{aligned} \mathscr{L} = \bar{\Psi}_1 \left(\hat{q} + g_S \langle S_0 \rangle + g_3 \langle S_3 \rangle + \frac{1}{\sqrt{3}} g_8 \langle S_8 \rangle \right) \Psi_1 + \\ \bar{\Psi}_2 \left(\hat{q} + g_S \langle S_0 \rangle - g_3 \langle S_3 \rangle + \frac{1}{\sqrt{3}} g_8 \langle S_8 \rangle \right) \Psi_2 + \bar{\Psi}_3 \left(\hat{q} + g_S \langle S_0 \rangle - \frac{2}{\sqrt{3}} g_8 \langle S_8 \rangle \right) \Psi_3 \end{aligned} \tag{5.45}$$

Let the VEVs of the scalar fields are $\langle S \rangle, \langle S_3 \rangle$ and $\langle S_8 \rangle$. The masses of the three quarks are

$$m_1 = -g_S \langle S_0 \rangle - g_3 \langle S_3 \rangle - \frac{1}{\sqrt{3}} g_8 \langle S_8 \rangle \tag{5.46}$$

$$m_2 = -g_S \langle S_0 \rangle + g_3 \langle S_3 \rangle - \frac{1}{\sqrt{3}} g_8 \langle S_8 \rangle \tag{5.47}$$

$$m_3 = -g_S \langle S_0 \rangle + \frac{2}{\sqrt{3}} g_8 \langle S_8 \rangle \tag{5.48}$$

The potential we receive is using the results from calculating the Feynman diagrams is

$$\begin{aligned} V[S_0, S_3, S_8] = -\frac{1}{2}\mu^2 S_0^2 - \frac{1}{2}\tilde{\mu}^2 (S_3^2 + S_8^2) + \frac{1}{2} g^2 S_0^4 + \frac{3}{4} g^2 (S_3^2 + S_8^2)^2 + \\ + 3g^2 S_0^2 (S_3^2 + S_8^2) - \sqrt{2} g^2 S_0 S_8^3 + 3\sqrt{2} g^2 S_0 S_3^2 S_8 \end{aligned} \tag{5.49}$$

For simplicity we introduce the variables $x = 3\sqrt{2} g \frac{S_0}{\tilde{\mu}}$, $y = \sqrt{3} g \frac{S_3}{\tilde{\mu}}$ and $z = 3g \frac{S_8}{\tilde{\mu}}$. Also, let $U = \frac{27 g^2}{\tilde{\mu}^4} V$ and $r^2 = \frac{\mu^2}{\tilde{\mu}^2}$ so that the new potential is a dimensionless quantity. Then the potential $U(x,y,z)$ is

$$\begin{aligned} U(x,y,z) = -\frac{3}{4} r^2 x^2 - \frac{3}{2}(3y^2 + z^2) + \frac{1}{24} x^4 + \frac{1}{4}(3y^2 + z^2)^2 + \frac{1}{2} x^2 (3y^2 + z^2) - \\ - \frac{1}{3} x z^3 + 3 x y^2 z \end{aligned} \tag{5.50}$$

Written in terms of the new variables the masses of the three quarks are defined as

$$m_1 = -\sqrt{-\frac{\tilde{\mu}^2}{18}}(x_0 + 3y_0 + z_0) \tag{5.51}$$

$$m_2 = -\sqrt{-\frac{\tilde{\mu}^2}{18}}(x_0 - 3y_0 + z_0) \tag{5.52}$$

$$m_3 = -\sqrt{-\frac{\tilde{\mu}^2}{18}}(x_0 - 2z_0) \tag{5.53}$$

where x_0, y_0 and z_0 are coordinates of extremal points of the potential U. In order to find where this function is minimal we need to solve a system of three equations

$$\partial_x U = 0,\ \partial_y U = 0, \text{ and } \partial_z U = 0. \tag{5.54}$$

We get the following system of equations

$$6\partial_x U = -9r^2 x_0 + x_0^3 + 6x_0(3y_0^2 + z_0^2) - 2z_0^3 + 18y_0^2 z_0 = 0 \tag{5.55}$$

$$\frac{1}{3}\partial_y U = -3y_0 + y_0(3y_0^2 + z_0^2) + x_0^2 y_0 + 2x_0 y_0 z_0 = 0 \tag{5.56}$$

$$\partial_z U = -3z_0 + z_0(3y_0^2 + z_0^2) + x_0^2 z_0 - x_0 z_0^2 + 3x_0 y_0^2 = 0 \tag{5.57}$$

We can immediately see that the second equation is solved by $y_0 = 0$. By inspection we see that this solution conserves isotopic symmetry. Since our main interest is to explore regimes where isotopic symmetry is broken we suppose $x_0 \neq 0$, $y_0 \neq 0$ and $z_0 \neq 0$. Then from 5.56 it follows that

$$y_0^2 = 1 - \frac{1}{3}(x_0 + z_0)^2 \tag{5.58}$$

Substituting for y_0^2 in 5.57 we get

$$x_0^2 + 2x_0 z_0 + 4z_0^2 - 3 = 0;\ z_0^{1,2} = \frac{1}{4}\left(-x_0 \pm \sqrt{3}\sqrt{4 - x_0^2}\right) \tag{5.59}$$

Substituting 5.58 and 5.59 in 5.55 we get

$$\pm\sqrt{12 - 3x_0^2}(1 - x_0^2) = 3(r^2 - 1)x_0. \tag{5.60}$$

From equation 5.56 and 5.57 we can show that

$$y_0^2 = z_0^2;\ y_0 = \pm z_0 \tag{5.61}$$

That relation between y and z is enough to show that in this general case proves that indeed we cannot break isospin symmetry within $SU(3)$ NJL model. Similar result for a $U(2)$ model is known from the papers by J. Preskill and S. Weinberg [46] and C. Vafa and E. Witten [47].

We consider each of the cases z_0^1 and z_0^2 separately (the two signs in 5.59). For the case of a positive sign we get

$$\sqrt{4-x^2}(1-x^2) = \sqrt{3}(r^2-1)x \tag{5.62}$$

The real solutions for x are when $|x| \leq 2$. We can solve graphically equation (34) where we can vary the value of r^2:

$$\text{For } t = x^2 \text{ and } a = r^2 - 1 \Rightarrow (4-t)(1-t)^2 = 3at \tag{5.63}$$

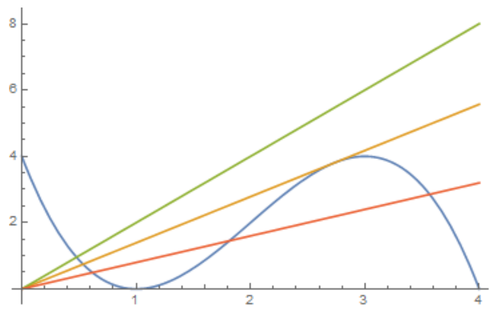


Fig. 5.1 Graphical solution of equation 5.63, where with blue we show the left hand side as a function of t, and with green, yellow and red we show the right hand side for three different values of a, or respectively r.

The boundary case of the tangent line is for a value of $a = a_0 = -3 + 2\sqrt{3}$. We have the first scenario of $a > a_0$, $a = a_0$ and $a < a_0$. For the first case we have one solution, for the second - two solutions and for the third case - three solutions. Each case will be considered in turn.

To find the analytical solutions of 5.63 we make the substitution $l = t - 2$. Then we can write it as

$$l^3 + 3(a-1)l + 6a - 2 = 0 \tag{5.64}$$

$$l_k = 2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) - k\frac{2\pi}{3}\right], \tag{5.65}$$

where $p = 3(a-1)$, $q = 6a-2$, and $k = 0,1,2$. Substituting these constants we finally get for t

$$t_k = 2\sqrt{1-a}\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{3a-1}{a-1}\frac{1}{\sqrt{1-a}}\right) - \frac{k\pi}{3}\right] + 2 \tag{5.66}$$

For values $a > a_0$ then

$$t' = \sqrt[3]{\sqrt{a^3+6a^2-3a}+3a-1} + \frac{a-1}{\sqrt[3]{\sqrt{a^3+6a^2-3a}+3a-1}} + 2 \tag{5.67}$$

From 5.58 we have that $y_0^2 = z_0^2$. Let $y_0 = z_0$. Then for the quark masses we get

$$m_1 = -\sqrt{-\frac{\tilde{\mu}^2}{18}}(x_0 + 4z_0) \tag{5.68}$$

$$m_2 = m_3 = -\sqrt{-\frac{\tilde{\mu}^2}{18}}(x_0 - 2z_0) \tag{5.69}$$

The electric charge is defined as

$$Q = T_3 - \beta T_8, \text{ where } \beta = -\frac{1}{\sqrt{3}},\ T_3 = \frac{1}{2}\lambda_3,\ T_8 = \frac{1}{2}\lambda_8 \tag{5.70}$$

For the quark charges we get

$$Q = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \tag{5.71}$$

meaning that the case $y_0 = z_0$ describes antiquarks and $q_1 = \bar{s}$, $q_2 = \bar{u}$ and $q_3 = \bar{d}$. In order to work with quarks we choose $y_0 = -z_0$. In this case the quark masses are

$$m_1 = -\frac{\tilde{\mu}}{3\sqrt{2}}(x_0 - 2z_0) = m_3 \tag{5.72}$$

$$m_2 = -\frac{\tilde{\mu}}{3\sqrt{2}}(x_0 + 4z_0) \tag{5.73}$$

In this case the parameter $\beta = \frac{1}{\sqrt{3}}$ and the charge matrix becomes

$$Q = T_3 + \frac{1}{\sqrt{3}} T_8 = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{5.74}$$

where $q_1 = u$, $q_2 = s$ and $q_3 = d$. It is evident that isospin symmetry is conserved within the choice of charge matrix, just as in the case of $SU(2)$ symmetry. Now we need to find a matrix acting on the fermion field such that it transforms q_2 into d and q_3 into s quark.

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} q_1' \\ q_2' \\ q_3' \end{pmatrix} \tag{5.75}$$

or $\Psi = T\Psi'$ as well as $\bar{\Psi} = \bar{\Psi}'T$. We also need to transform the meson fields such that the new Lagrangian is equivalent to 5.20. This way we obtain that $S_0 = S_0'$ and

$$\lambda_3 S_3 + \lambda_8 S_8 = \left(\frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8 \right) S_3' + \left(\frac{\sqrt{3}}{2}\lambda_3 - \frac{1}{2}\lambda_8 \right) S_8' \tag{5.76}$$

or written in matrix form

$$\begin{pmatrix} S_0 \\ S_3 \\ S_8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} S_0' \\ S_3' \\ S_8' \end{pmatrix} \tag{5.77}$$

Similar equation can be written also for the variables x, y and z. Their general transformation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \tag{5.78}$$

In the special case when $y_0 = -z_0$ we have

$$\begin{pmatrix} x_0' \\ y_0' \\ z_0' \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \\ -2z_0 \end{pmatrix} \tag{5.79}$$

We also need to check the equivalence of $U(x,y,z)$ and $U(x',y',z')$ which is done by straightforward substitution of $x = x'$, $y = \frac{y'+z'}{2}$ and $z = \frac{3y'-z'}{2}$. Indeed,

$$U(x',y',z') = U(x,y,z) \tag{5.80}$$

We saw that when $(x,y,z) \to (x',y',z')$ and $y_0 = -z_0$ then $y'_0 = 0$. We have to show that under these conditions the extremal points of $U(x',y',z')$ have the same coordinates as $U(x,y,z)$ when $x \neq 0$, $y \neq 0$ and $z \neq 0$. We get the following system of equations

$$-9r^2x'_0 + x'^3_0 + 6x'_0z'^2_0 - 2z'^3_0 = 0 \tag{5.81}$$

$$-3 + z'^2_0 + x^2_0 - x'_0z'_0 = 0 \tag{5.82}$$

Eliminating z'_0 from the system we get

$$\pm(1 - x'^2_0)\sqrt{4 - x'^2_0} = \sqrt{3}(r^2 - 1)x'_0 \tag{5.83}$$

which has the same solutions as (34). Therefore, the transformation (x_0, y_0, z_0) into $(x'_0, 0, z'_0)$ gives identical results.

The next task is to prove that the Hessian for $U(x_0, y_0, z_0)$ is the same as for the matrix $U(x'_0, 0, z'_0)$. We do that by constructing the Hessian matrix from its definition in the case of $U(x'_0, 0, z'_0)$. Then we show that $A^T H(x_0, y_0, z_0)A = H(x'_0, y'_0, z'_0)$. By definition

$$H(x,y,z) = \begin{pmatrix} U_{xx} & U_{xy} & U_{xz} \\ U_{yx} & U_{yy} & U_{yz} \\ U_{zx} & U_{zy} & U_{zz} \end{pmatrix}, \tag{5.84}$$

where the indices mean differentiation with respect to a variable. When $y'_0 = 0$

$$H(x'_0, y'_0, z'_0) = \begin{pmatrix} -\frac{3}{2}r^2 + \frac{1}{2}x'^2_0 + z'^2_0 & 0 & 2x'_0z'_0 - z'^2_0 \\ 0 & -9 + 3(z'_0 + x'_0)^2 & 0 \\ 2x'_0z'_0 - z'^2_0 & 0 & -3 + 3z'^2_0 + x'^2_0 - 2x'_0z'_0 \end{pmatrix} \tag{5.85}$$

Also,

$$H(x_0, y_0, z_0) = \begin{pmatrix} -\frac{3}{2}r^2+\frac{1}{2}x_0^2+3y_0^2+z_0^2 & 6x_0y_0+6y_0z_0 & 3y_0^2+2x_0z_0-z_0^2 \\ 6x_0y_0+6y_0z_0 & -9+3x_0^2+27y_0^2+6x_0z_0+3z_0^2 & 6x_0y_0+6y_0z_0 \\ 3y_0^2+2x_0z_0-z_0^2 & 6x_0y_0+6y_0z_0 & -3+x_0^2+3y_0^2-2x_0z_0+3z_0^2 \end{pmatrix} \tag{5.86}$$

Then we calculate the product $A^T H(x_0, y_0, z_0)A$ and use the transformation equations to go into the primed coordinate system. In the case when $y'_0 = 0$ we have that $x_0 = x'_0$, $y_0 = \frac{1}{2}z'_0$ and $z_0 = -\frac{1}{2}z'_0$. Then we indeed get that

$$A^T H(x_0, y_0, z_0)A = H(x'_0, y'_0, z'_0) \tag{5.87}$$

We would now like to extend the $SU(3)$ model to its complete version so that we are able to derive the mass spectrum of all scalar mesons.

$$\mathscr{L} = \bar{\Psi}\hat{q}\Psi + g_S\bar{\Psi}\Psi S_0 + \sum_{i=1}^{8} g_i\bar{\Psi}\lambda_i\Psi S_i - \frac{g_S^2}{2G_0}S_0^2 - \sum_{i=1}^{8}\frac{g_i^2}{2G'_0}S_i^2 \tag{5.88}$$

Having all results from the Feynman diagrams we can write the interacting potential and derive the mass matrix

$$\begin{aligned} V[S_0, S_1 ... S_8] = {} & \frac{\mu^2}{2}S_0^2 + \frac{\tilde{\mu}^2}{2}\sum_{a=1}^{8} S_a^2 + \frac{g^2}{2}S_0^4 + \frac{3}{4}g^2\sum_{a=1}^{8}\left(S_a^2\right)^2 \\ & +3g^2S_0^2\sum_{a=1}^{8} S_a^2 - \sqrt{2}g^2S_0S_8^3 + 3\sqrt{\frac{3}{2}}g^2S_0S_3\left(S_4^2 + S_5^2 - S_6^2 - S_7^2\right) + \\ & + \frac{3\sqrt{2}}{2}g^2S_0S_8\left(2S_1^2 + 2S_2^2 + 2S_3^2 - S_4^2 - S_5^2 - S_6^2 - S_7^2\right) + \\ & +3\sqrt{6}g^2S_0\left(S_1S_4S_6 + S_1S_5S_7 - S_2S_4S_7 + S_2S_5S_6\right) \end{aligned} \tag{5.89}$$

To find the extremal points of this potential we need to solve the system of equations

$$\frac{\partial V}{\partial S_a} = 0 \tag{5.90}$$

for $a = 0...8$. Since the vacuum does not violate strangeness we put $\langle S_k\rangle = \langle S_{4...7}\rangle = 0$. We also proved that setting $\langle S_3\rangle = 0$ does not violate isotopic symmetry. Also, only diagonal terms give contributions to mass, therefore we can also set $\langle S_{1,2}\rangle = 0$. The non-trivial equations defining the extremal points of V are

$$\frac{\partial V}{\partial S_0} = 2g^2\langle S_0\rangle^3 + 6g^2\langle S_0\rangle\langle S_8\rangle^2 - \sqrt{2}g^2\langle S_8\rangle^3 + \mu^2\langle S_0\rangle = 0 \tag{5.91}$$

$$\frac{\partial V}{\partial S_8} = 6g^2\langle S_0\rangle^2\langle S_8\rangle - 3\sqrt{2}g^2\langle S_0\rangle\langle S_8\rangle^2 + 3g^2\langle S_8\rangle^3 + \tilde{\mu}^2\langle S_8\rangle = 0 \tag{5.92}$$

$$\frac{\partial^2 V}{\partial S_0^2} = m_0^2 = 4g^2\langle S_0\rangle^2 + \sqrt{2}g^2\frac{\langle S_8\rangle^3}{\langle S_0\rangle} \tag{5.93}$$

$$\frac{\partial^2 V}{\partial S_{1,2,3}^2} = m_1^2 = 9\sqrt{2}g^2\langle S_0\rangle\langle S_8\rangle \tag{5.94}$$

$$\frac{\partial^2 V}{\partial S_{4,5,6,7}^2} = m_2^2 = \tilde{\mu}^2 + 6g^2\langle S_0\rangle^2 - 3\sqrt{2}g^2\langle S_0\rangle\langle S_8\rangle + 3g^2\langle S_8\rangle^2 = 0 \tag{5.95}$$

$$\frac{\partial^2 V}{\partial S_8^2} = m_8^2 = -3\sqrt{2}g^2\langle S_0\rangle\langle S_8\rangle + 6g^2\langle S_8\rangle^2 \tag{5.96}$$

$$\frac{\partial^2 V}{\partial S_i \partial S_j} = 0, \text{ where } i \neq j \neq 0, 8 \tag{5.97}$$

$$\Delta_{08} = 12g^2\langle S_0\rangle\langle S_8\rangle - 3\sqrt{2}g^2\langle S_8\rangle^2 \tag{5.98}$$

In the case of $SU(2)$ theory, quarks acquire mass through spontaneous chiral symmetry breaking and pions are massless pseudo-Goldstone bosons. In order for them to acquire mass we need to explicitly break the chiral symmetry by introducing a massive quark in the Lagrangian. We observed $U(2)$ breaks into $U(1) \times SU(2)$ and $\langle S_3\rangle = 0$ preserves isotopic symmetry. Since $SU(2) \subset SU(3)$ then we set again $\langle S_3\rangle = 0$. In the present case we have $U(3) \to U(1) \times SU(3) \to U(1) \times SU(2) \times U(1)$ which is broken by the mass of S_8. We should expect to have 4 Goldstone bosons which is exactly the case - we have all κ mesons with zero mass. In order for them to acquire mass we need to explicitly introduce massive quarks in the $SU(3)$ lagrangian, where the u and d quark have the same mass m_0 and the s quark has a different mass m_s:

$$-\mathscr{M}_0\bar{\Psi}\Psi = -\begin{pmatrix} m_0 & 0 & 0 \\ 0 & m_0 & 0 \\ 0 & 0 & m_s \end{pmatrix}\bar{\Psi}\Psi \tag{5.99}$$

The structure of κ mesons is $(q\bar{s})$ and because of the presence of s quark we should expect them to be heavier than $S_{3,4,5}$ which are the $a_0(980)$ mesons composed of $(q\bar{q})$. This is not the case and the mass spectrum is inverted which we will later prove.

The mass matrix for S_0 and S_8 becomes

$$\mathscr{M}_{08}^2 = \begin{pmatrix} m_0^2 & \Delta_{08} \\ \Delta_{08} & m_8^2 \end{pmatrix} = -\tilde{\mu}^2 \begin{pmatrix} \frac{2}{9}x_0^2 + \frac{2}{9}\frac{z_0^3}{x_0} & \frac{2\sqrt{2}}{3}x_0z_0 - \frac{\sqrt{2}}{3}z_0^2 \\ \frac{2\sqrt{2}}{3}x_0z_0 - \frac{\sqrt{2}}{3}z_0^2 & -\frac{1}{3}x_0z_0 + \frac{2}{3}z_0^2 \end{pmatrix} \tag{5.100}$$

To get physical particle masses we require $\det\left[\mathscr{M}^2_{08}\right] > 0$ and $\mathrm{Tr}\left[\mathscr{M}^2_{08}\right] > 0$. The first condition reads

$$\frac{1}{3x_0}(x_0^3+z_0^3)(2z_0-x_0)-z_0(2x_0-z_0)^2>0 \tag{5.101}$$

After applying 5.59 this condition simplifies to

$$x_0^4-2x_0^2-2<0 \tag{5.102}$$

This inequality has the solutions $0<x_0^2<1+\sqrt{3}$ or $0<t<1+\sqrt{3}$.We already saw that the solutions of the equations 5.55, 5.56 and 5.57 when the variables x, y and z are all different from 0 we do not get solution where isotopic symmetry is broken. Next we turn to exploring the case when $y_0=0$. We can write 5.55 and 5.57 in terms of x, y and z variables to get

$$\begin{aligned}\frac{\partial U}{\partial x}&=-\frac{3}{2}r^2x_0+\frac{x_0^3}{6}+x_0z_0^2-\frac{z_0^3}{3}=0\\ \frac{\partial U}{\partial z}&=-3z_0+z_0^3+x_0^2z_0-x_0z_0^2=0\end{aligned} \tag{5.103}$$

From 5.103 we get the relations

$$z_0^{1,2}=\frac{x_0\pm\sqrt{12-3x_0^2}}{2} \tag{5.104}$$

$$z_0^2=3+x_0z_0-x_0^2 \tag{5.105}$$

Simplifying the equations 5.103 we get

$$2z_0(x_0^2-1)-x_0^3+4x_0=3r^2x_0 \tag{5.106}$$

Case I: $z_0=z_1=\frac{x+\sqrt{12-3x_0^2}}{2}$. Then

$$(x_0^2-1)\sqrt{4-x_0^2}=\sqrt{3}(r^2-1)x_0 \tag{5.107}$$

We can distinguish four separate cases: Ia $r^2\geq 1, 1\leq x_0\leq 2$; Ib $r^2\geq 1, -1\leq x_0\leq 0$. When $x_0>0$ $x_0^2\geq 1$ and when $x_0<0$ then $x_0^2\leq 1$. Ic $0\leq r^2\leq 1$, $0\leq x_0\leq 1$ and Id $0\leq r^2\leq 1$, $-2\leq x_0\leq -1$.

Case II considers $z_2=\frac{x_0-\sqrt{12-3x_0^2}}{2}$

$$(x_0^2-1)\sqrt{4-x_0^2}=\sqrt{3}(1-r^2)x_0 \tag{5.108}$$

IIa $r^2 \geq 1$, $0 \leq x_0 \leq 1$; IIb $r^2 \geq 1$, $-2 \leq x_0 \leq -1$; IIc $0 \leq r^2 \leq 1$, $1 \leq x_0 \leq 2$; IId $0 \leq r^2 \leq 1$, $-1 \leq x_0 \leq 0$.

The reality of the theory requires that the absolute value of the s quark must be bigger than the absolute value of the mass of u and d quark. We have that when $y_0 = 0$ then $m_s \sim x_0 - 2z_0$ and $m_u \sim x_0 + z_0$. Then in case I we get

$$\sqrt{12-3x_0^2} \geq \left| \frac{3x_0+\sqrt{12-3x_0^2}}{2} \right| \tag{5.109}$$

This inequality has the case when the term in the brackets is positive or negative. If it is positive we get

$$\sqrt{4-x_0^2} \geq -\sqrt{3}x_0 \tag{5.110}$$

which is always met when x_0 is positive. If $-2 \leq x_0 < 0$ then

$$4-x_0^2 \geq 3x_0^2 \tag{5.111}$$

leading to $x_0 \leq 1$ leading to $x_0 \in [-1,2]$. For this interval of x_0 we can open the brackets in 5.109 without change in the signs to get:

$$\sqrt{12-3x_0^2} \geq 3x_0 \tag{5.112}$$

For positive x_0 we get $x_0 \leq 1$ and for negative x_0 $-2 \leq x_0 \leq 1$ which corresponds to solutions Ib and Ic. For the inequality 5.110 the term in the brackets may also be negative. Then we get

$$\sqrt{3}x_0+\sqrt{4-x_0^2} \leq 0, \text{ or } -2 \leq x_0 \leq -1 \tag{5.113}$$

Then the term in the brackets comes out with a negative sign, from where we get

$$x_0^2 \leq 3, \text{ or} -\sqrt{3} \leq x_0 \leq -1 \tag{5.114}$$

which is analogous solution Id. We repeat this whole analysis for the case II, where $z_0 = z_2$.

$$\sqrt{12-3x_0^2} \geq \left| \frac{3x_0-\sqrt{12-3x_0^2}}{2} \right| \tag{5.115}$$

One solution is

$$\sqrt{3}x_0 \geq \sqrt{4-x_0^2}, \text{ or } 1 \leq x_0 \leq 2 \tag{5.116}$$

Within this interval we have that

$$\sqrt{12-3x_0^2} \geq x_0, \text{ from where } 1 \leq x_0 \leq \sqrt{3} \tag{5.117}$$

or finally $1 \leq x_0 \leq \sqrt{1+\sqrt{3}}$. Again, we have the scenario where the term in the brackets is negative, then $-1 \leq x_0 \leq 1$ and within this interval we reproduce the solutions IIa,d. This ends the quark masses condition analysis. Now we turn to discuss the trace condition on the square mass matrix 5.98:

$$\mathrm{Tr}\left[\mathcal{M}_{08}^2\right] = \frac{2}{9}\left(\frac{z_0^3}{x_0} + x_0^2\right) + \frac{1}{3}\left(2z_0^2 - x_0 z_0\right) > 0 \tag{5.118}$$

This conditions is always met regardless of the connection between x_0 and z_0 that we use.

The final condition that we apply to our model comes from the mass m_1. In 5.94 essentially we have the product of x_0 and z_0. Since we require to have real particle masses we want this product to be positive which happens only when both x_0 and z_0 are simultaneously positive or negative. This restriction leaves us only with two final cases:

$$0 \leq x_0 \leq 1 \text{ and } 0 \leq r^2 \leq 1 \tag{5.119}$$

$$-1 \leq x_0 \leq 0 \text{ and } 0 \leq r^2 \leq 1 \tag{5.120}$$

We can also derive the same result graphically. Plotting the functions describng the quark masses we get the following graph

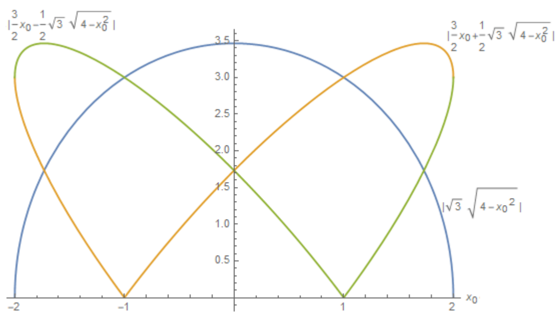


Fig. 5.2 Plot of the functions describing the constituent quark masses.

We see that case I function in the negative region goes to zero at $x_0 = -1$ and the mass of the s quark is positive. This is not a physical situation since at that point there must be symmetry in the quark masses. The mass of the s quark is bigger than the mass of the u and d quark only in the interval [-1;1]. In the negative section we conclude that $z_0 = z_0^2$. Following the same logic we get that in the positive section $z_0 = z_0^1$. So, finally

$$\begin{aligned} x_0 \in [-1;0] : |m_s| = \sqrt{3}\sqrt{4-x_0^2} \text{ and } z_0 = z_0^2 = \frac{3}{2}x_0 - \frac{1}{2}\sqrt{3}\sqrt{4-x_0^2} \\ x_0 \in [0,1] : |m_s| = \sqrt{3}\sqrt{4-x_0^2} \text{ and } z_0 = z_0^1 = \frac{3}{2}x_0 + \frac{1}{2}\sqrt{3}\sqrt{4-x_0^2} \end{aligned} \tag{5.121}$$

Chapter 6

Conclusion

In this thesis we laid out all the theoretical basis that was covered in the theoretical minimum courses, which are obligatory for the successful completion of the PhD degree. We explored mass relations between meson excitations in $U(1)$ and $SU(2)$ Nambu–Jona-Lasinio models with massless quarks and managed to derive a novel relation between the meson masses for the $SU(2)$ theory:

$$R \equiv \frac{2m_{\rho'}^2 - m_{\rho'}m_{\rho} + 2m_{\rho}^2}{3m_{b_1}^2} = 1 \; . \tag{6.1}$$

We also showed that introducing explicit symmetry breaking quark mass in the $U(1)$ theory does not introduce any change in the mass relations. In the case of a massive quark we get

$$R \equiv \frac{2m_{\phi}^2 - m_{\phi}m_{\phi'} + 2m_{\phi'}}{3m_h^2} = 1 \tag{6.2}$$

Then we explored the $U(2)$ and $U(3)$ scalar NJL models. We observed that isotopic symmetry is not violated and we also showed that the $U(3)$ model with massless quarks is not a physical theory since it predicts that the κ mesons are massless Goldstone bosons, which is not what we observe experimentally. These results are published in the following papers (in chronological order):

1. M. Chizhov and M. Naydenov. Isospin-invariant Nambu – Jona-Lasinio model with complete set of spin-1 excitations.*AIP Conference Proceedings*,2075,2019.

2. M. Chizhov and M. Naydenov. Novel Mass Relation Among Spin-1 Hadron Resonances and Quark Masses (In Bulgarian).*Annuaire de l'Université de Sofia "St. Kliment Ohridski"*, 10.

3. M. Naydenov and M. Chizhov. On the Spontaneous Breaking of Isotopic Symmetry in the Nambu and Jona-Lasinio Model (In Russian), *PEPAN*, 17, 2, 2020.

The material from the first paper was reported on the 10th Jubilee International Conference of the Balkan Physical Union, 26-30.08.2018, Sofia, Bulgaria. All other results have been presented on the yearly reports in front of the Atomic Physics Division at the Faculty of Physics, Sofia University. As a part of my degree I participated in a school organized by JINR, Dubna, held in Borovets, 14-17.05.2018.

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